8.3 Volumes

What you will learn about . . .

- Volume As an Integral → $V = \int \text{area}$
- Square Cross Sections
- Circular Cross Sections
- Cylindrical Shells
- Other Cross Sections

and why . . .

The techniques of this section allow us to compute volumes of certain solids in three dimensions.

DEFINITION Volume of a Solid

The **volume of a solid** of known integrable cross-section area $A(x)$ from $x = a$ to $x = b$ is the integral of A from a to b,

$$V = \int_a^b A(x)\,dx.$$

volume area

How to Find Volume by the Method of Slicing

1. Sketch the solid and a typical cross section.
2. Find a formula for $A(x)$.
3. Find the limits of integration. a, b
4. Integrate $A(x)$ to find the volume.

Square Cross Sections

Let us apply the volume formula to a solid with square cross sections.

EXAMPLE 1 A Square-Based Pyramid

A pyramid 3 m high has congruent triangular sides and a square base that is 3 m on each side. Each cross section of the pyramid parallel to the base is a square. Find the volume of the pyramid.

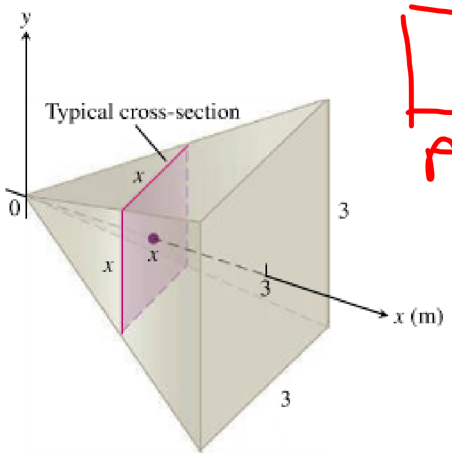
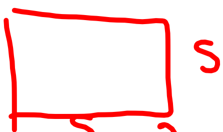


Figure 8.17 A cross section of the pyramid in Example 1.



$A = s^2$

$$V = \int_0^3 s^2\,ds$$

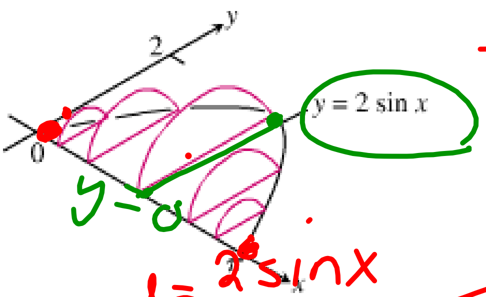
$$= \frac{s^3}{3}\Big|_0^3$$

$$(9) - (0)$$

$$\boxed{V = 9\,m^3}$$

EXAMPLE 7 A Mathematician's Paperweight

A mathematician has a paperweight made so that its base is the shape of the region between the x-axis and one arch of the curve $y = 2\sin x$ (linear units in inches). Each cross section cut perpendicular to the x-axis (and hence to the xy-plane) is a semicircle whose diameter runs from the x-axis to the curve. (Think of the cross section as a semicircular fin sticking up out of the plane.) Find the volume of the paperweight.



$d = 2\sin x$

$r = \sin x$

$A = \frac{1}{2}\pi r^2$

$A = \frac{1}{2}\pi \sin^2 x$

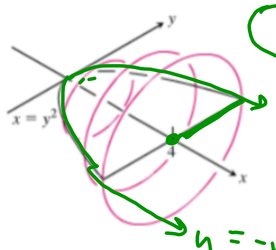
$V = \int_0^{\pi} \frac{1}{2}\pi \sin^2 x \, dx$

Calculator
{ math 9
alpha window 4

$V = 2.467 \text{ in}^3$

2. The solid lies between planes perpendicular to the x-axis at $x = 0$ and $x = 4$. The cross sections perpendicular to the x-axis between these planes run from $y = -\sqrt{x}$ to $y = \sqrt{x}$.

(a) The cross sections are circular disks with diameters in the xy-plane.



$\pi r^2 = A$

$r = \sqrt{x}$

$y = \sqrt{x}$

$\pi(\sqrt{x})^2 = A$

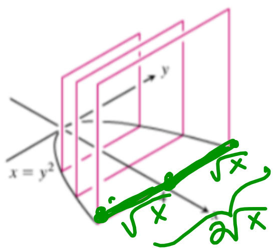
$\pi x = A$

$y = -\sqrt{x}$

$$V = \int_0^4 \pi x \, dx$$

$$= 8\pi u^3$$

(b) The cross sections are squares with bases in the xy-plane.



$\sqrt{x}$ $\sqrt{x}$ $2\sqrt{x}$

$s = 2\sqrt{x}$

$A = s^2$

$A = (2\sqrt{x})^2$

$A = 4x$

$$A = \int_0^4 4x \, dx$$

$$\frac{64}{3} u^3$$

$y = \sqrt{x}$

$y = -\sqrt{x}$

(c) The cross sections are squares with diagonals in the xy-plane.

$a^2 + b^2 = c^2$

$s^2 + s^2 = (2\sqrt{x})^2$

$2s^2 = 4x$

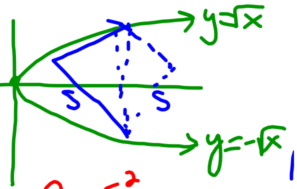
$\sqrt{s^2} = \sqrt{2x}$

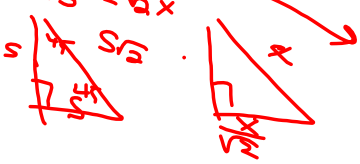
$2\sqrt{x}$, s, 45

$s = \frac{2\sqrt{x}}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}$

$s = \frac{2\sqrt{2x}}{2}$

$s = \sqrt{2x}$





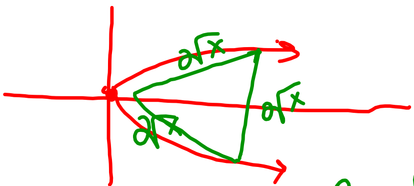
$A = s^2$

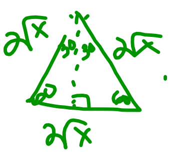
$A = (\sqrt{2x})^2$

$A = 2x$

$$V = \int_0^4 2x \, dx$$

(d) The cross sections are equilateral triangles with bases in the xy-plane.



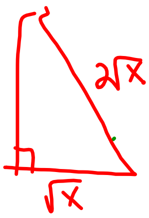


$A = \frac{1}{2} bh$

$\frac{1}{\cancel{2}} \cancel{2}\sqrt{x} \cdot \sqrt{3x}$

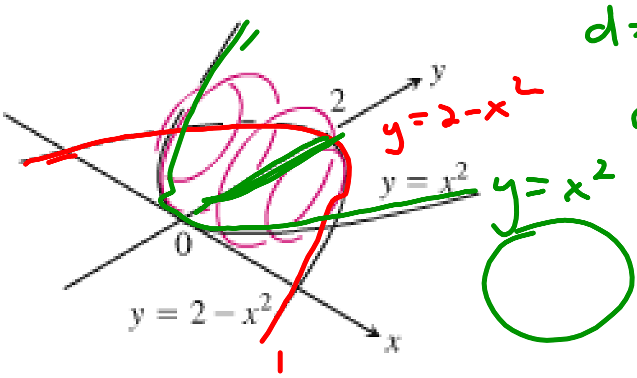
$A = x\sqrt{3}$

$\sqrt{3x} = h$



$$V = \int_0^4 x\sqrt{3} \, dx$$

4. The solid lies between planes perpendicular to the x-axis at $x = -1$ and $x = 1$. The cross sections perpendicular to the x-axis are circular disks whose diameters run from the parabola $y = x^2$ to the parabola $y = 2 - x^2$.



$d = 2 - x^2 - x^2$

$d = 2 - 2x^2$

$r = 1 - x^2$

$A = \pi r^2$

$\pi(1-x^2)^2$

$V = \pi \int_{-1}^{1} (1-x^2)^2 \, dx$

$(1-x^2)(1-x^2)$

$V = \pi \int_{-1}^{1} 1 - 2x^2 + x^4 \, dx$

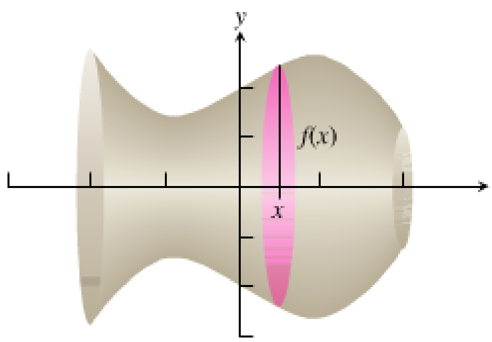
$\boxed{V = \frac{16\pi}{15} u^3}$

Circular Cross Sections

The only thing that changes when the cross sections of a solid are circular is the formula for $A(x)$. Many such solids are **solids of revolution,** as in the next example.

EXAMPLE 2 A Solid of Revolution

The region between the graph of $f(x) = 2 + x\cos x$ and the x-axis over the interval $[-2, 2]$ is revolved about the x-axis to generate a solid. Find the volume of the solid.



Homework 8.3:
Day 1 1-7 odd,39 (cross sections)
Day 2 11,16,19,22(circles and washers)
Day 3 9,23,27,41(rotate around the y-axis)