# Section 8.2

Areas in the Plane

## What you'll learn about

- Area Between Curves
- Area Enclosed by Intersecting Curves
- Boundaries with Changing Functions
- Integrating with Respect to *y*
- Saving Time with Geometric Formulas

...and why

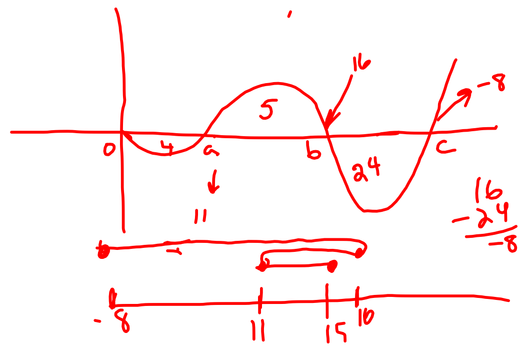
The techniques of this section allow us to compute areas of complex regions of the plane.

$$\int_0^{24} 3.9 - 2.4\sin\left(\frac{\pi t}{12}\right)dt$$

$$3.9t + 2.4\cos\left(\frac{\pi t}{12}\right)\frac{12}{\pi}\Bigg|_0^{24} \text{ Kw-hrs}$$

$$\left(3.9(24)+2.4\overbrace{\cos(2\pi)}^{1}\cdot\frac{12}{\pi}\right)-\left(0+2.4\cos(0)\frac{12}{\pi}\right)$$

$$\boxed{93.6 \text{ Kw hrs}}$$




$$\int_0^{10} 27.08e^{t/25}\,dt \quad \text{yrs} \quad \frac{\text{bil of barrels}}{\text{yr}}$$

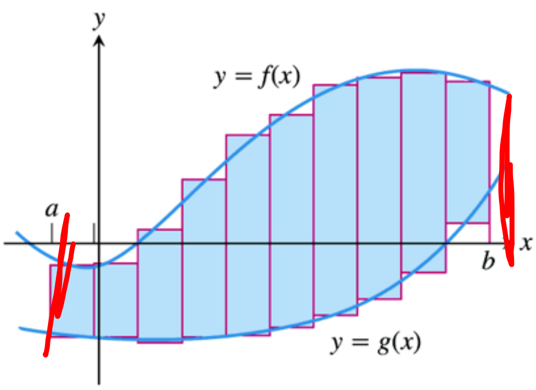
$$27.08e^{t/25}\cdot 25\Big|_0^{10}$$

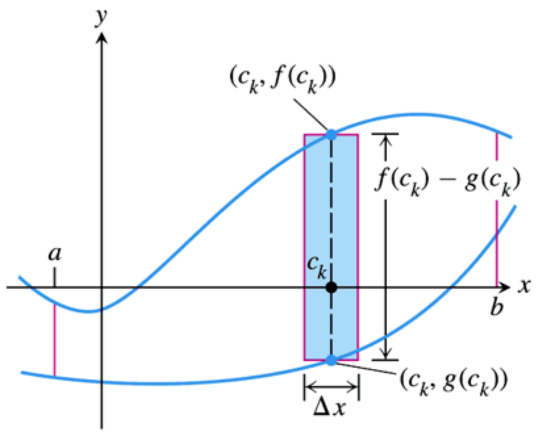
$$\left(27.08e^{10/25}\cdot 25\right)-\left(27.08e^{0}\cdot 25\right)$$

332.965 bil of barrels

# Area Between Curves

Partition the region into vertical strips of equal width $\Delta x$. Each rectangle has area $[f(c_k)-g(c_k)]\Delta x$ for some $c_k$ in its respective subinterval. Approximate the area of each region with the Riemann sum $\sum[f(c_k)-g(c_k)]\Delta x$.




The limit of these sums as $\Delta x \to 0$ is $\int_a^b [f(x)-g(x)]\,dx$.

top - bottom

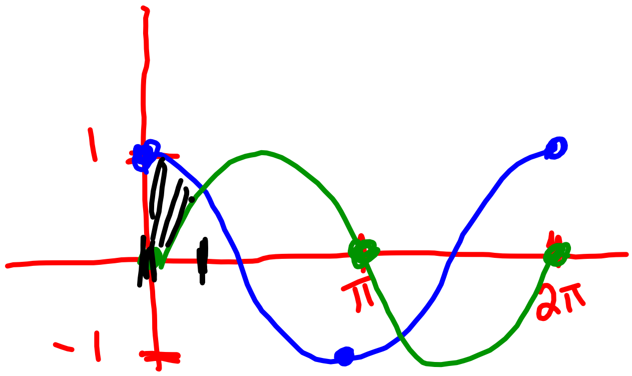
# Area Between Curves

If $f$ and g are continuous with $f(x) \geq g(x)$ throughout $[a,b]$, then the area between the curves $y = f(x)$ and $y = g(x)$ from $a$ to $b$ is the integral of $[f - g]$ from $a$ to $b$,

$$\boxed{A = \int_a^b [f(x) - g(x)]\,dx}$$

# Example Applying the Definition

Find the area of the region between $y = \cos x$ and $y = \sin x$ from $x = 0$ to $x = \frac{\pi}{4}$




$$\int_0^{\pi/4} \cos x - \sin x \, dx$$

$$\sin x + \cos x \Big|_0^{\pi/4}$$

$$\left(\sin\frac{\pi}{4} + \cos\frac{\pi}{4}\right) - \left(\sin 0 + \cos 0\right)$$

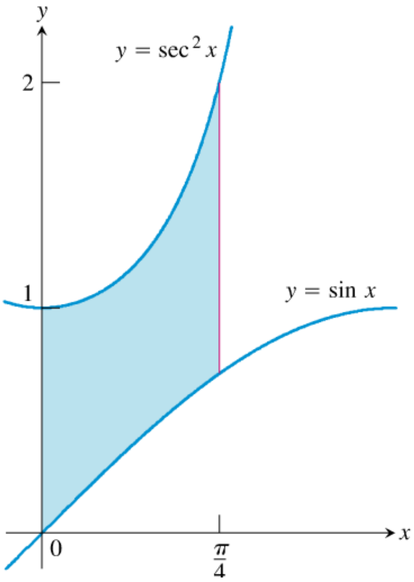
$$\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - (0 + 1)$$

$$\frac{2\sqrt{2}}{2}$$

$$\boxed{\sqrt{2} - 1}\ u^2$$

**EXAMPLE 1** **Applying the Definition**

Find the area of the region between $y = \sec^2 x$ and $y = \sin x$ from $x = 0$ to $x = \pi/4$.




$$\int_0^{\pi/4} \sec^2 x - \sin x \, dx$$

$$\tan x + \cos x \Big|_0^{\pi/4}$$

$$\left(\tan\tfrac{\pi}{4} + \cos\tfrac{\pi}{4}\right) - \left(\tan 0 + \cos 0\right)$$

$$\left(1 + \frac{\sqrt{2}}{2}\right) - \left(0 + 1\right)$$

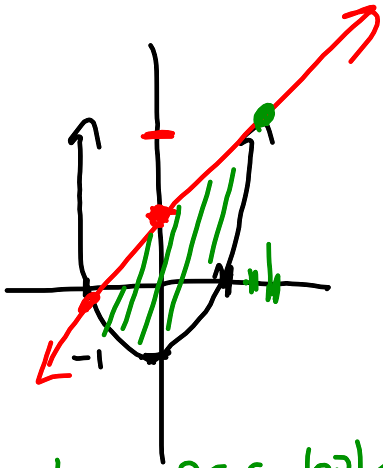
$$1 + \frac{\sqrt{2}}{2} - 1$$

$$\boxed{\frac{\sqrt{2}}{2}\, u^2}$$

# Example Area of an Enclosed Region

Find the area of the region enclosed by the parabola $y = x^2 - 1$ and $y = x + 1$. Graph the curves to view the region.

$m = 1$
$b = 1$



$$x^2 - 1 = x + 1$$
$$-x - 1 \quad -x - 1$$
$$x^2 - x - 2 = 0$$
$$(x-2)(x+1) = 0$$
$$x = 2 \quad x = -1$$

line - parabola

$$\int_{-1}^{2} \underbrace{x+1-(x^2-1)}_{x+1-x^2+1}\,dx$$

$$\int_{-1}^{2} -x^2 + x + 2\,dx$$

$$\left. -\frac{x^3}{3} + \frac{x^2}{2} + 2x \right|_{-1}^{2}$$

$$\left(-\frac{8}{3} + 2 + 4\right) - \left(\frac{1}{3} + \frac{1}{2} - 2\right)$$

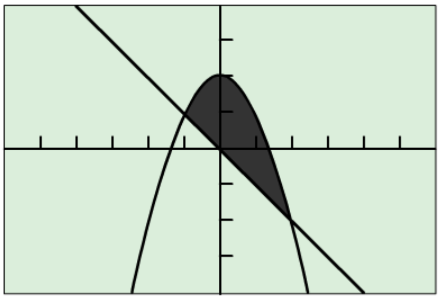
$$-\frac{8}{3} + 6 - \frac{1}{3} - \frac{1}{2} + 2$$

$$8 - 3 - \frac{1}{2}$$

$$5 - \frac{1}{2} = \boxed{4.5u^2}$$

## EXAMPLE 2 Area of an Enclosed Region

Find the area of the region enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$.



[–6, 6] by [–4, 4]

## Example Using Geometry

Find the area of the region enclosed by the graphs of $y = \sqrt{x+1}$, left 1
$y = x - 1$ and the $x$-axis.

$\sqrt{x+1}^2 = (x-1)^2$

$(x-1)(x-1)$

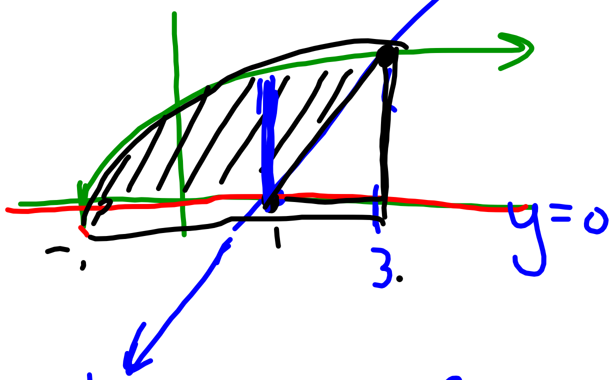
$x+1 = x^2 - 2x + 1$

$-x-1 \quad -x \quad -1$

$0 = x^2 - 3x$

$x(x-3) = 0$

$x = 0 \quad x = 3$




$\int_{-1}^{1} \sqrt{x+1} - 0\,dx + \int_{1}^{3} \sqrt{x+1} - (x-1)\,dx$

alpha window 4

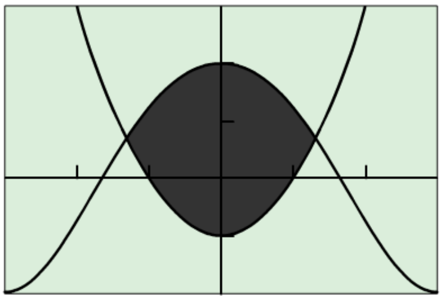
$\boxed{3.333u^2}$

$\int_{-1}^{3} \sqrt{x+1}\,dx - \Delta$

$\frac{1}{2}(2)(2)$

## EXAMPLE 3 Using a Calculator

Find the area of the region enclosed by the graphs of $y = 2\cos x$ and $y = x^2 - 1$.



[–3, 3] by [–2, 3]

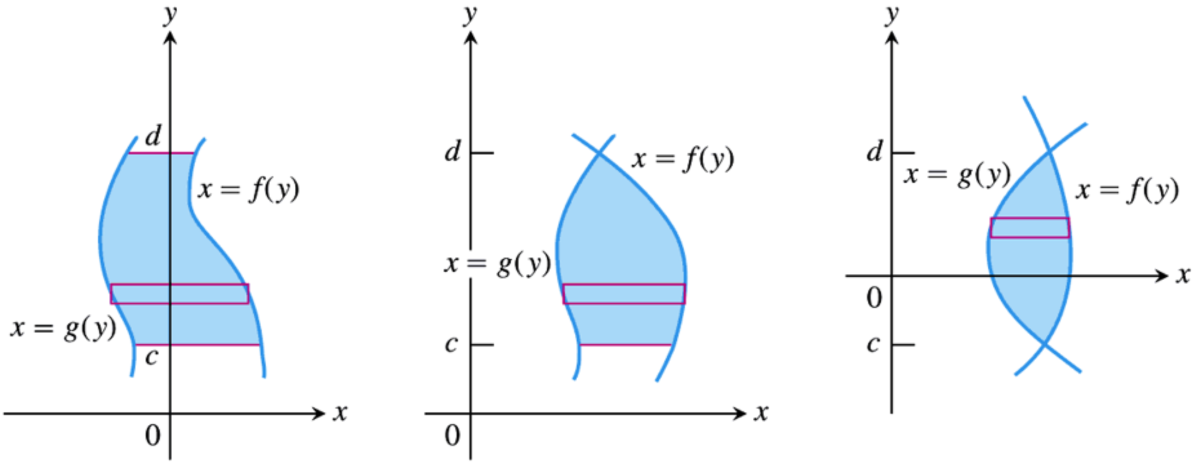
## EXAMPLE 4 Finding Area Using Subregions

Find the area of the region $R$ in the first quadrant that is bounded above by $y = \sqrt{x}$ and below by the $x$-axis and the line $y = x - 2$.

# Integrating with Respect to y

If the boundaries of a region are more easily described by functions of $y$, use horizontal approximating rectangles.

For regions like these




use this formula

$$A = \int_c^d [f(y) - g(y)]\,dy.$$

# Example Integrating with Respect to $y$

**EXAMPLE 4 Finding Area Using Subregions**

Find the area of the region $R$ in the first quadrant that is bounded above by $y = \sqrt{x}$ and below by the $x$-axis and the line $y = x - 2$.

Homework 8.2:

Day 1: 3,6,9,15,18,21,30

Day 2: 12,24,27,36,39,42