Section 7.4

Homework Day 1:
3-18 by 3

Exponential Growth and Decay

$A = Pe^{rt}$ continuous

$A = P\left(1 + \frac{r}{n}\right)^{nt}$ monthy daily, quart

What you'll learn about

- Separable Differential Equations ✓
- Law of Exponential Change ✓
- Continuously Compounded Interest ✓
- Modeling Growth with Other Bases
- Newton's Law of Cooling

… and why

Understanding the differential equation $\frac{dy}{dx} = ky$ gives us new insight into exponential growth and decay.

Separable Differential Equation

A differential equation of the form $\frac{dy}{dx} = f(y)g(x)$ is called **separable**. We **separate the variables** by writing it in the form

$$\frac{1}{f(y)}dy = g(x)dx.$$

The solution is found by antidifferentiating each side with respect to its thusly isolated variable.

Example Solving by Separation of Variables

Solve for y if $\frac{dy}{dx} = x^2 y^2$ and $y = 3$ when $x = 0$.

$dx \qquad dx$

$$\frac{dy}{y^2} = \frac{x^2 y^2 dx}{y^2}$$

$$\frac{1}{y^2}dy = x^2 dx$$

$$y^{-2}dy = x^2 dx \qquad (0,3)$$

$$\frac{y^{-1}}{-1} = \frac{x^3}{3} + C$$

$$-\frac{1}{y} = \frac{x^3}{3} + C$$

$$-\frac{1}{3} = \frac{0^3}{3} + C$$

$$C = -\frac{1}{3}$$

$$-\frac{1}{y} = \underbrace{\frac{x^3}{3} - \frac{1}{3}}$$

$$-\frac{1}{y} = \frac{x^3 - 1}{3}$$

$$(-1)\ -y = \frac{3}{x^3-1}\ (-1)$$

$$\boxed{y = \frac{-3}{x^3-1}}$$

EXAMPLE 1 **Solving by Separation of Variables**

Solve for y if $dy/dx = (xy)^2$ and $y = 1$ when $x = 1$.

$$\frac{dy}{dx} = x^2 y^2$$

$$\frac{1}{y^2}\,dy = x^2\,dx$$

$$y^{-2}\,dy = x^2\,dx$$

$$\frac{y^{-1}}{-1} = \frac{x^3}{3} + C$$

$(1,1)$

$$-\frac{1}{y} = \frac{x^3}{3} + C$$

$$-1 = \frac{1}{3} + C$$

$$-\frac{1}{3} \qquad -\frac{1}{3}$$

$$-\frac{4}{3} = C$$

$$-\frac{1}{y} = \frac{x^3}{3} - \frac{4}{3}$$

$$-\frac{1}{y} = \frac{x^3 - 4}{3}$$

$$-y = \frac{3}{x^3 - 4}$$

$$\boxed{y = \frac{-3}{x^3 - 4}}$$

The Law of Exponential Change

If y changes at a rate proportional to the amount present (that is, if $\frac{dy}{dt} = ky$), and if $y = y_0$ when $t = 0$, then

$x=0$

$$\boxed{y = y_0 e^{kt}} \approx A = Pe^{rt}$$

The constant k is the **growth constant** if $k > 0$ or the **decay constant** if $k < 0$.

$$\frac{dy}{dx} = ky$$

$$\frac{1}{y}\,dy = k\,dx$$

$$\ln|y| = kx + c$$

$$\log_e y = kx + c$$

$$e^{kx+c} = y$$

$$y = e^{kx} \cdot \underbrace{e^c}_{\#} \rightarrow 2.71828^{\#}$$

$\log_5 25 = 2$

$5^2 = 25$

$x^2 \cdot x^4 = x^{2+4}$

$= x^6$

$$y = \underbrace{e^{k \cdot 0}}_{1} \cdot e^c \longrightarrow \boxed{y = y_0 e^{kt}}$$

$$y = e^c$$

$$y = y_0$$

Continuously Compounded Interest

If the interest is added continuously at a rate proportional to the amount in the account, you can model the growth of the account with the initial value problem:

Differential equation: $\boxed{\frac{dA}{dt} = rA}$ $\frac{dy}{dt} = ky$

Initial condition: $A(0) = A_o$

The amount of money in the account after t years at an annual interest rate r:

$\boxed{A(t) = A_o e^{rt}}$.

Continuously Compounded Interest

Suppose that A_0 dollars are invested at a fixed annual interest rate r (expressed as a decimal). If interest is added to the account k times a year, the amount of money present after t years is

$$A(t) = A_0\left(1 + \frac{r}{k}\right)^{kt}.$$

Example Compounding Interest Continuously

Suppose you deposit $500 in an account that pays 5.3% annual interest. How much will you have 4 years later if the interest is **(a)** compounded continuously? **(b)** compounded monthly?

$P = 500$
$r = .053$
$t = 4$

a) continuously

$A = Pe^{rt}$

$A = 500e^{.053(4)}$

$A = \$618.07$

b) monthly $n = 12$

$A = P\left(1 + \frac{r}{n}\right)^{nt}$

$A = 500\left(1 + \frac{.053}{12}\right)^{12 \cdot 4}$

$A \approx \$617.79$

EXAMPLE 2 **Compounding Interest Continuously**

Suppose you deposit $800 in an account that pays 6.3% annual interest. How much will you have 8 years later if the interest is **(a)** compounded continuously? **(b)** compounded quarterly?

$P = 800$

$r = .063$

$t = 8$

a) $A = Pe^{rt}$

$A = 800e^{.063(8)}$

$1324.26

b) $n = 4$

$800\left(1 + \frac{.063}{4}\right)^{4 \cdot 8}$

$1319.07

double: $\frac{2P}{P} = \frac{Pe^{.063t}}{P}$

$2 = e^{.063t}$

$\log_e 2 = .063t$

$\frac{\ln 2}{.063} = \frac{.063t}{.063}$

$t = \frac{\ln 2}{.063}$