
6.4 Fundamental Theorem of Calculus

What you will learn about . . .

- Fundamental Theorem, Part 1 ✖
- Graphing the Function $\int_a^x f(t) dt$
- Fundamental Theorem, Part 2
- Area Connection
- Analyzing Antiderivatives Graphically

and why . . .

The Fundamental Theorem of Calculus is a triumph of mathematical discovery and the key to solving many problems.

THEOREM 4 The Fundamental Theorem of Calculus, Part 1

If f is continuous on $[a, b]$ then the function

$$F(x) = \int_a^x f(t) dt$$

has a derivative at every point x in $[a, b]$, and

$$\frac{dF}{dx} = \frac{d}{dx} \int_a^x f(t) dt = f(x)$$

*a more indepth proof is on pages 298.

$F(x)$ = original!
 $f(x)$ = derivative of $F(x)$

EXAMPLE 1 Applying the Fundamental Theorem

Find

$$\frac{d}{dx} \int_{-\pi}^{e^x} \cos t \, dt$$

by using the Fundamental Theorem.

 $\cos x$

$$\frac{d}{dx} \int_0^x \frac{1}{1+t^2} dt$$

EXAMPLE 2 The Fundamental Theorem with the Chain RuleFind $\frac{dy}{dx}$ if $y = \int_1^{x^2} \cos t \, dt$.

The image shows a handwritten solution in red and green ink. The original problem is written in red: $\frac{d}{dx} \int_1^{x^2} \cos t \, dt$. The derivative operator $\frac{d}{dx}$ is circled in red. A green arrow points from the upper limit x^2 to the integrand $\cos t$. Below this, the result of the chain rule is written in green: $\cos(x^2) \cdot 2x$. This result is enclosed in a green rectangular box, with the final answer $2x \cos(x^2)$ written inside the box.

$$\frac{d}{dx} \int_1^{x^2} \cos t \, dt$$
$$\cos(x^2) \cdot 2x$$
$$2x \cos(x^2)$$

EXAMPLE 3 Variable Lower Limits of IntegrationFind $\frac{dy}{dx}$

(a) $y = \int_x^5 3t \sin t \, dt$

The handwritten solution shows the derivative of the integral $y = \int_x^5 3t \sin t \, dt$. The derivative is calculated as $-\frac{d}{dx} \int_x^5 3t \sin t \, dt$. The derivative of the integral with respect to the lower limit x is $-3x \sin x$.

$$-\frac{d}{dx} \int_x^5 3t \sin t \, dt = -3x \sin x$$

Find dy/dx : (b) $y = \int_{2x}^{x^2} \frac{1}{2+e^t} dt$

$$\frac{d}{dx} \int_{2x}^{x^2} \frac{1}{2+e^t} dt = \frac{d}{dx} \left(\int_0^{x^2} \frac{1}{2+e^t} dt - \int_0^{2x} \frac{1}{2+e^t} dt \right)$$

$$= \frac{1}{2+e^{x^2}} \cdot 2x - \frac{1}{2+e^{2x}} \cdot 2$$

$$\boxed{\frac{2x}{2+e^{x^2}} - \frac{2}{2+e^{2x}}}$$

EXAMPLE 4 Constructing a Function with a Given Derivative and ValueFind a function $y = f(x)$ with derivative

$$\frac{dy}{dx} = \tan x$$

that satisfies the condition $f(3) = 5$.

$$f(x) = y$$

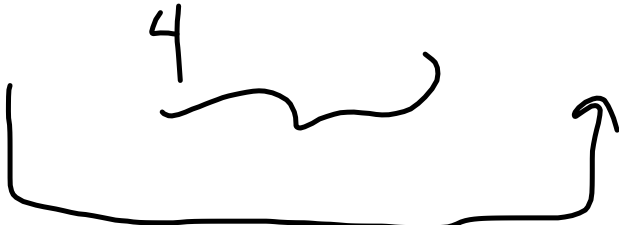
$\tan x$

$$y = \int_3^x \tan t \, dt + 5$$

$0 + 5$

$$y = 5$$

Example: Construct a function with $dy/dx=e^x$ satisfying the condition that $f(4)=-7$.

$$y = \int_4^x e^t dt - 7$$


Example: #10 Find dy/dx if: $\frac{dy}{dx} = \int \cot(3t) dt$

$$\cot(3x^2) \cdot 2x$$

$$\frac{dy}{dx} = 2x \cot(3x^2)$$

Example: #16 Find dy/dx if:

$$y = \int_{5x^2}^{25} \frac{t^2 - 2t + 9}{t^3 + 6} dt$$

$$\frac{d}{dx} y = \frac{d}{dx} \int_{5x^2}^{25} \frac{t^2 - 2t + 9}{t^3 + 6} dt$$

$$\frac{dy}{dx} = - \frac{(5x^2)^2 - 2(5x^2) + 9}{(5x^2)^3 + 6} \cdot 10x$$

$$= \frac{-10x}{1} \cdot \frac{25x^4 - 10x^2 + 9}{125x^6 + 6}$$

$$= \frac{-10x(25x^4 - 10x^2 + 9)}{125x^6 + 6}$$

$$\frac{-250x^5 + 100x^3 - 90x}{125x^6 + 6}$$

Homework 6.4 Day 1: 3-24 by 3