Real Zeros of Polynomial Functions

- Long Division and the Division Algorithm
- Remainder and Factor Theorems
- Synthetic Division
- Rational Zeros Theorem
- Upper and Lower Bounds

Section 2-4

Worksheet

$$\begin{array}{r} 7534 \\ 9\,\overline{)\,67809} \\ \underline{63} \\ 48 \\ \underline{45} \\ 30 \\ \underline{27} \\ 39 \\ \underline{36} \\ \boxed{3} \end{array}$$

Quotient

Divisor

Dividend

Remainder

$67809 \div 9 = 7534\frac{3}{9}$

$7534\frac{1}{3}$

$67809 = 9(7534)+3$

Division Algorithm for Polynomials

Let $f(x)$ and $d(x)$ be polynomials with the degree of f greater than or equal to the degree of d, and $d(x) \neq 0$. Then there are unique polynomials $q(x)$ and $r(x)$, called the **quotient** and **remainder**, such that

$$f(x) = d(x) \cdot q(x) + r(x)$$

where either $r(x) = 0$ or the degree of r is less than the degree of d.

$$f(x) = d(x) \cdot q(x) + r(x)$$

dividend = (divisor)(quotient) + remainder

Divide f(x) by d(x) and write a summary statement in polynomial form and fraction form.

$f(x) = x^2 - 2x + 3$ $d(x) = x - 1$

$$\begin{array}{r|l} & x \quad -1 \\ \hline x-1 & x^2 - 2x + 3 \\ & x^2 - x \\ \hline & -x + 3 \\ & -x + 1 \\ \hline \end{array}$$

$$\frac{f(x)}{d(x)} = x - 1 + \frac{2}{x-1}$$

$(x-1)(x-1)+2 = x^2-2x+\overset{2}{3}$

$x^2-2x+1+2$

$x^2-2x+3 = x^2-2x+3$ ✓

Divide f(x) by d(x) and write a summary statement in polynomial form and fraction form.

$f(x) = 2x^3 - 2x^2 + x + 3$ $d(x) = x^2 + x + 3$

$$2x - 4$$

$$x^2 + x + 3 \,\overline{)\, 2x^3 - 2x^2 + x + 3}$$

$$\underline{2x^3 + 2x^2 + 6x}$$

$$-4x^2 - 5x + 3$$

$$\underline{-4x^3 - 4x - 12}$$

$$-x + 15$$

$$\frac{f(x)}{a(x)} = 2x - 4 + \frac{-x+15}{x^2+x+3}$$

$$(2x-4)(x^2+x+3) - x + 15 = f(x)$$

Using Polynomial Long Division

Use long division to find the quotient and remainder when $2x^4 - x^3 - 2$ is divided by $2x^2 + x + 1$. Write a summary statement in both polynomial and fraction form.

$$2x^2 + x + 1 \,\overline{)\, 2x^4 - x^3 + 0x^2 + 0x - 2}$$

Remainder Theorem

If polynomial $f(x)$ is divided by $x - k$, then the remainder is $r = f(k)$.

Factor Theorem

A polynomial function $f(x)$ has a factor $x - k$ if and only if $f(k) = 0$.

$(x-k)$ remainder $=0$

$x-k$ is a factor

k is a zero ex: $x-2=0$

$x=2$

Using the Remainder Theorem

Find the remainder when $f(x) = 3x^2 + 7x - 20$ is divided by...

a) $x - 2$

$r(x)=f(2)$ $\quad r(x) = 3(2)^2+7(2)-20$

$12+14-20=\boxed{6}$

b) $x + 1$

$(x - -1)$ $f(-1)= 3(-1)^2+7(-1)-20$

$3-7-20=-24$

c) $x + 4$

$f(-4)=3(-4)^2+7(-4)-20$

Are any of the above factors of f(x)?

$48-28-20=0$

$\boxed{x+4 \text{ is a factor}}$

Fundamental Connections for Polynomial Functions

For a polynomial function f and a real number k the following statements are equivalent:

1. $x = k$ is a solution (or root) of the equation $f(x) = 0$
2. k is a zero of the function f.
3. k is an x-intercept of the graph of $y = f(x)$.
4. $x - k$ is a factor of $f(x)$.

Using Synthetic Division

Divide $2x^3 - 3x^2 - 5x - 12$ by $x - 3$ using synthetic division and write a summary statement.

Divide using synthetic division, and write a summary statement in fraction form.

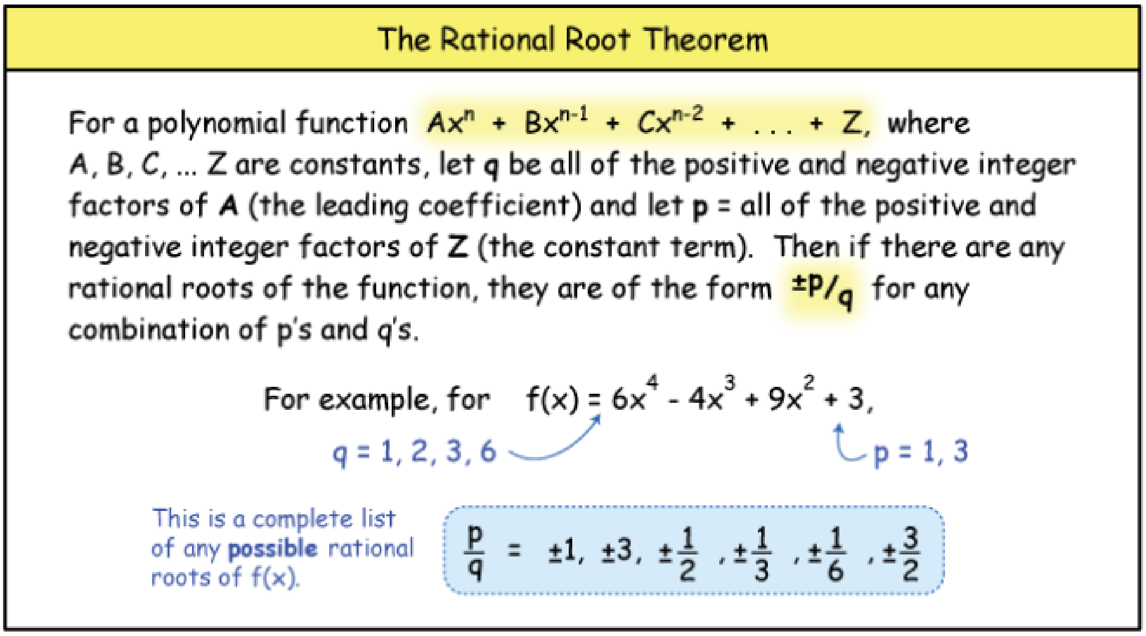
$$\frac{x^3 - 5x^2 + 3x - 2}{x + 1}$$

Find all the zero of $8x^3 - 30x^2 + 13x + 30$

given $x - 2$ is a factor

Find all the zero of $2x^3 - x^2 - 41x - 20$

given $2x + 1$ is a factor



The Rational Root Theorem

For a polynomial function $Ax^n + Bx^{n-1} + Cx^{n-2} + \dots + Z$, where A, B, C, ... Z are constants, let **q** be all of the positive and negative integer factors of **A** (the leading coefficient) and let **p** = all of the positive and negative integer factors of **Z** (the constant term). Then if there are any rational roots of the function, they are of the form $\pm p/q$ for any combination of p's and q's.

For example, for $f(x) = 6x^4 - 4x^3 + 9x^2 + 3$,

$q = 1, 2, 3, 6$ $p = 1, 3$

This is a complete list of any **possible** rational roots of f(x).

$$\frac{p}{q} = \pm 1, \pm 3, \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{6}, \pm\frac{3}{2}$$

Finding the Rational Zeros

Find the rational zeros of $f(x) = x^3 - 3x^2 + 1$

Finding the Rational Zeros

Find the rational zeros of $f(x) = 3x^3 + 4x^2 - 5x - 2$

Use the Rational Zeros Theorem to write a list of all potential rational zeros. Then determine which ones, if any, are zeros.

$$f(x) = 6x^3 - 5x - 1$$

Find all real zeros of the function, finding exact values whenever possible. Identify each zero as rational or irrational.

$$f(x) = 2x^4 - 7x^3 - 2x^2 - 7x - 4$$