BC Calculus
Chapter 10 Review

Name________________

1) Write the first four terms of the series $\sum_{n=2}^{\infty} \frac{x^n}{3n-1}$.

$\frac{x^2}{5} + \frac{x^3}{8} + \frac{x^4}{11} + \frac{x^5}{14}$

2) Tell whether the series $\sum_{n=1}^{\infty} 4\left(\frac{2}{5}\right)^n$ converges or diverges. If it converges, find its sum.

Since $r=\frac{2}{5}$ and $-1<\frac{2}{5}<1$ $\therefore$ Convergent

$g_1=\frac{8}{5}$ $S_\infty = \frac{\frac{8}{5}}{1-2/5} = \frac{\frac{8}{5}}{3/5} = \frac{8}{3}$

3) Given that $1 - x + x^2 + \cdots + (-x)^n$ is a power series representation for $\frac{1}{1+x}$, find a power series representation for $\frac{x^3}{1+x^2}$.

$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots (-(x^2))^n$

$x^3\left(\frac{1}{1+x^2}\right) = x^3 - x^5 + x^7 - x^9 + \cdots x^3(-x^2)^n$

4) Find the Taylor polynomial of order 3 generated by $f(x) = \sin 2x$ at $x = \frac{\pi}{4}$.

$f(\frac{\pi}{4}) = \sin\frac{\pi}{2} = 1$

$f'(x) = 2\cos 2x$

$f'(\frac{\pi}{4}) = 2\cos\frac{\pi}{2} = 0$

$f''(x) = -4\sin 2x$

$f''(\frac{\pi}{4}) = -4\sin\frac{\pi}{2} = -4$

$f'''(\frac{\pi}{4}) = 0$

$f(a) + \frac{f'(a)(x-a)^1}{1!} + \frac{f''(a)(x-a)^2}{2!} + \frac{f'''(a)(x-a)^3}{3!}$

$1 + \cancel{\frac{0(x-\frac{\pi}{4})^1}{1!}} + \frac{-4(x-\frac{\pi}{4})^2}{2!} + \cancel{\frac{0(x-\frac{\pi}{4})^3}{3!}}$

$1 - 2(x-\frac{\pi}{4})^2$

5) Let f be a function that has derivatives of all orders for all real numbers. Assume $f(0) = 5, f'(0) = -3$, $f''(0) = 8$, and $f'''(0) = 24$. Write the third order Taylor polynomial for f at $x = 0$ and use it to approximate $f(0.4)$.

$5 + \frac{-3(x-0)}{1!} + \frac{8(x)^2}{2!} + \frac{24(x)^3}{3!}$

$5 - 3x + 4x^2 + 4x^3$

$5 - 3(.4) + 4(.4)^2 + 4(.4)^3$

6) The Maclaurin series for $f(x)$ is

$1 + 2x + \frac{3x^2}{2} + \frac{4x^3}{6} + \cdots + \frac{(n+1)x^n}{n!} + \cdots$.

(a) Find $f''(0)$.

(b) Let $g(x) = xf'(x)$. Write the Maclaurin series for $g(x)$.

(c) Let $h(x) = \int_0^x f(t)\,dt$. Write the Maclaurin series for $h(x)$.

$\frac{4x^3}{3!}$ $\frac{12x^2}{3!}$

a) $f'(x) = 2 + 3x + 2x^3$

$f''(x) = 3 + \text{—}$

b) $f'(x) = 2 + 3x + 2x^2$

$xf'(x) = 2x + 3x^2 + 2x^3 + \cdots + \frac{(n+1)x^n}{(n-1)!}$

c) $x + x^2 + \frac{x^3}{2} + \frac{x^4}{6} + \cdots + \frac{x^{n+1}}{n!}$

7) Find the Taylor polynomial of order 4 for $f(x) = e^{-x^2}$ at $x = 0$ and use it to approximate $f(0.3)$.

$f(0) = 1$

$f'(x) = -2xe^{-x^2}$ $f'(0) = 0$

$f''(x) = -2e^{-x^2} + (2x)(-2x)e^{-x^2}$ $f''(0) = -2$

8) Determine the convergence or divergence of each series. Identify the test (or tests) you use.

(a) $\sum_{n=2}^{\infty} \frac{(2n)!}{(n-1)3^n}$ (b) $\sum_{n=1}^{\infty} \frac{(n^2+3n-4)}{n!}$

(c) $\sum_{n=1}^{\infty} \left(1 + \frac{1}{2n}\right)^{3n}$

9) Determine whether each series converges absolutely, converges conditionally, or diverges.

(a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n \ln n}$ (b) $\sum_{n=1}^{\infty} \frac{\sin n}{n^3}$

(c) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2n)!}{(2n-1)!}$

$\lim_{n\to\infty} \frac{a_{k+1}}{a_k}$

$\sum_{n=0}^{\infty} \frac{(+2)^{3n}}{8^n}$

$\sum_{n=0}^{\infty} \left(\frac{8}{8}\right)^n$

$\sum_{n=0}^{\infty} (1)^n$

10) Find the interval of convergence of the series $\sum_{n=0}^{\infty} \frac{(4x-3)^{3n}}{8^n}$ and, within this interval, the sum of the series as a function of x.

$-1 < \frac{(4x-3)^3}{8} < 1$

$-8 < (4x-3)^3 < 8$

$-2 < 4x-3 < 2$

$1 < 4x < 5$

$\frac{1}{4} < x < \frac{5}{4}$

$S_\infty = \frac{1}{1 - \frac{(4x-3)^3}{8}}$

$S_\infty = \frac{a_1}{1-r}$

11) Find the interval of convergence of the series $\sum_{n=1}^{\infty} \frac{3^n(x-2)^n}{\sqrt{n+2}\cdot 2^n}$.

$\lim_{n\to\infty} \frac{3^{n+1}(x-2)^{n+1}}{\sqrt{n+3}\cdot 2^{n+1}} \cdot \frac{\sqrt{n+2}\cdot 2^n}{3^n(x-2)^n}$

$\lim_{n\to\infty} \frac{3(x-2)}{2} \cdot \frac{\sqrt{n+2}}{\sqrt{n+3}}$

ratio: $\frac{3}{2}(x-2)$

$-1 < \frac{3(x-2)}{2} < 1$

$-2 < 3(x-2) < 2$

$-\frac{2}{3} < x-2 < \frac{2}{3}$

$\frac{4}{3} \leq x \leq \frac{8}{3}$

12) Find the radius of convergence of each power series.

(a) $\sum_{n=0}^{\infty} \frac{(3x)^n}{\sqrt{n}}$ (b) $\sum_{n=1}^{\infty} \frac{n^2(2x-3)^n}{6^n}$

a) $-1 < 3x < 1$

$-\frac{1}{3} \leq x < \frac{1}{3}$

r.o.c $= \frac{1}{3}$

$\boxed{\text{a) ratio: } \lim_{n\to\infty} \frac{(3x)^{n+1}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{(3x)^n}}$

$\lim_{n\to\infty} \left(\frac{\sqrt{n}}{\sqrt{n+1}}\right) 3x$

$-1 < 3x < 1$

b)

$-1 < \frac{2x-3}{6} < 1$

$-6 < 2x-3 < 6$

$-3 < 2x < 9$

$-\frac{3}{2} < x < \frac{9}{2}$

r.o.c $= 3$

Find the interval of convergence of the series $\sum_{n=1}^{\infty} \frac{3^n(x-2)^n}{\sqrt{n+2}\cdot 2^n}$. $4/3$ $8/3$

$\sum_{n=1}^{\infty} \frac{3^n\left(-\frac{2}{3}\right)^n}{\sqrt{n+2}\; 2^n}$

$\sum_{n=1}^{\infty} \frac{3^n\left(\frac{2}{3}\right)^n}{\sqrt{n+2}\; 2^n}$ 1^n

$= \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+2}}$

$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+2}}$ $\frac{1}{(n+2)^{\frac{1}{2}}}$

$\sum^{\infty} (n+2)^{-\frac{1}{2}}$

$\lim_{n\to\infty} \frac{1}{\sqrt{n+2}}$

$\lim_{n\to\infty} \frac{1}{\sqrt{n+3}} \cdot \frac{\sqrt{n+2}}{1}$

8. Determine the convergence or divergence of each series. Identify the test (or tests) you use.

(a) $\sum_{n=2}^{\infty} \frac{(2n)!}{(n-1)3^n}$ (b) $\sum_{n=1}^{\infty} \frac{(n^2+3n-4)}{n!}$

(c) $\sum_{n=1}^{\infty} \left(1+\frac{1}{2n}\right)^{3n}$

a) $\sum_{n=2}^{\infty} \frac{(2n)!}{(n-1)3^n}$

n^{th} Term

$\lim_{n\to\infty} \frac{(2n)!}{(n-1)3^n} = \infty$ Div.

$\therefore$ by n^{th} Term Test it Div.

$\lim_{n\to\infty} \frac{(2n+2)!}{n \cdot 3^{n+1}} \cdot \frac{(n-1)3^n}{(2n)!} = \lim_{n\to\infty} \left(\frac{1}{3}\right) \frac{(2n+2)(2n+1)(n-1)}{n} = \infty$

b) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+3n-4}{n!}$

nth Term
$\lim_{n\to\infty} a_n = 0$

~~geo~~ ~~p-ser~~

Int / Ratio / comp

$$\lim_{n\to\infty} \frac{(n+1)^2+3(n+1)-4}{(n+1)!} \cdot \frac{n!}{n^2+3n-4}$$

$$\lim_{n\to\infty} \frac{1}{n+1} \cdot \frac{n^2+5n}{n^2+3n-4} = 0$$

$\therefore$ Since $L = 0$
$-1 < 0 < 1$ by Ratio Test Series conv.

c) $\sum_{n=1}^{\infty} \left(1+\frac{1}{2n}\right)^{3n}$

n^{th} Term

$\lim_{n\to\infty} \left(1+\frac{1}{2n}\right)^{3n}$

$\lim_{n\to\infty} (1+0)^{3n} = \infty$

by N^{th} Term Series diverges.

9. Determine whether each series converges absolutely, converges conditionally, or diverges.

(a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n \ln n}$ (b) $\sum_{n=1}^{\infty} \frac{\sin n}{n^3}$

(c) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2n)!}{(2n-1)!}$

9) a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n \ln n}$

Conv. Abs

$\sum_{n=1}^{\infty} \frac{1}{n \ln n}$

nth Term $\lim_{n\to\infty} \frac{1}{n \ln n} = 0$

~~Geo~~

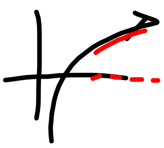
~~P-ser~~

Ratio / Comp / Inte

$\int_1^{\infty} \frac{1}{n \ln n} = \lim_{b\to\infty} \int_1^b \frac{1}{n \ln n}$

$\int \frac{1}{n \ln n} dn \quad u = \ln n$

$du = \frac{1}{n} dn$



$= \lim_{b\to\infty} \left[\ln(\ln(n)) \right]_1^b$

$\int \frac{1}{u} du$

$= \lim_{b\to\infty} \left[\ln \ln(b) - \ln \ln(1) \right]$

$\ln u$

∞

$\ln(\ln n)$

$= \infty$ ∴ by Integral Test Series Div.

Cond Conv?

$\boxed{\sum_{n=1}^{\infty} \frac{(-1)^n}{n \ln n}}$ by A.S.T.

1) pos/decr. ✓

$\sum_{n=1}^{\infty} \frac{1}{n \ln n}$

2) $\lim_{n\to\infty} a_n = 0$

∴ Series conv. by A.S.T.

Thus the $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n \ln n}$ is conditionally conv.

b) $\displaystyle\sum_{n=1}^{\infty} \left|\frac{\sin n}{n^3}\right|$ $\quad -1 \leftrightarrow 1 \quad \rightarrow \frac{1}{n^3}$

Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$ conv. by p-series where $p>1$, $p=3$

and $\displaystyle\frac{\sin n}{n^3} \le \frac{1}{n^3} \ \forall n$ then $\displaystyle\sum_{n=1}^{\infty} \frac{\sin n}{n^3}$ is conv.

Absolutely conv.

c) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2n)!}{(2n-1)!}$

$2n \cdot \cancel{2n-1} \cdot \cancel{2n-2} \cdots \cancel{2\cdot 1}$

$\cancel{2n-1} \cdot \cancel{2n-2} \cdots \cancel{2\cdot 1}$

$\sum_{n=1}^{\infty} (-1)^{n-1} \boxed{(2n)}$

, n^{th} Term Test $\lim_{n\to\infty}(a_n) \neq 0$

$\therefore$ by n^{th} term test the series D.V.