A particle travels with velocity

$$v(t) = (t - 2)\sin t \ \text{m/sec}$$

for $0 \le t \le 4$ sec.

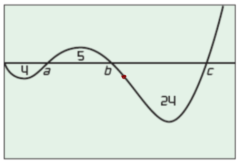
(a) What is the particle's displacement? ≈ -1.44952 meters

(b) What is the total distance traveled? ≈ 1.91411 meters

$\int_0^4 (t-2)(\sin t)\,dt$

$\int_0^4 |(t-2)\sin(t)|\,dt$

In Exercises 12–16, a particle moves along the x-axis (units in cm). Its initial position at $t = 0$ sec is $x(0) = 15$. The figure shows the graph of the particle's velocity $v(t)$. The numbers are the *areas* of the enclosed regions.



12. What is the particle's displacement between $t = 0$ and $t = c$? -23 cm

13. What is the total distance traveled by the particle in the same time period? 33 cm

14. Give the positions of the particle at times a, b, and c. a: 11 b: 16 c: -8

15. Approximately where does the particle achieve its greatest positive acceleration on the interval $[0, b]$?

16. Approximately where does the particle achieve its greatest positive acceleration on the interval $[0, c]$?

7.2 Areas in the Plane

What you'll learn about

- Area Between Curves
- Area Enclosed by Intersecting Curves
- Boundaries with Changing Functions
- Integrating with Respect to y
- Saving Time with Geometric Formulas

. . . and why

The techniques of this section allow us to compute areas of complex regions of the plane.

DEFINITION Area Between Curves

If f and g are continuous with $f(x) \ge g(x)$ throughout $[a, b]$, then the **area between the curves $y = f(x)$ and $y = g(x)$ from a to b** is the integral of $[f - g]$ from a to b,

$$A = \int_a^b [f(x) - g(x)]\,dx.$$

EXAMPLE 1 Applying the Definition

Find the area of the region between $y = \sec^2 x$ and $y = \sin x$ from $x = 0$ to $x = \pi/4$.

$\int_0^{\pi/4} \sec^2 x\,dx - \int_0^{\pi/4} \sin x\,dx$

$\int_0^{\pi/4} \sec^2 x - \sin x\,dx$

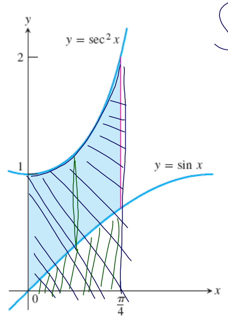


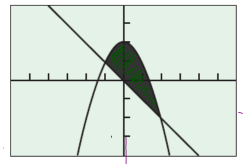
Figure 7.6 The region in Example 1.

Area Enclosed by Intersecting Curves

When a region is enclosed by intersecting curves, the intersection points give the limits of integration.

EXAMPLE 2 Area of an Enclosed Region

Find the area of the region enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$.



[−6, 6] by [−4, 4]

Figure 7.8 The region in Example 2.

$2 - x^2 = -x$

$0 = x^2 - x - 2$

$(x-2)(x+1)$

$x = 2 \quad x = -1$

$\int_{-1}^{2} (2-x^2) - (-x)\,dx$

$\int_{-1}^{2} -x^2 + x + 2\,dx$

$-\frac{1}{3}x^3 + \frac{1}{2}x^2 + 2x \Big]_{-1}^{2}$

$-\frac{1}{3}(2)^3 + \frac{1}{2}(2)^2 + 2(2) - \left[-\frac{1}{3}(-1)^3 + \frac{1}{2}(-1)^2 + 2(-1)\right]$

4.5

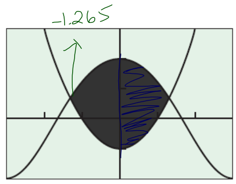
$y = 2 - x^2 + 2 = -x^2 + 4$ $\quad y = -x + 2$

$\int_{-1}^{2} (-x^2+4) - (-x+2)\,dx$

$\int_{-1}^{2} -x^2 + x + 2$

EXAMPLE 3 Using a Calculator

Find the area of the region enclosed by the graphs of $y = 2\cos x$ and $y = x^2 - 1$.



[−3, 3] by [−2, 3]

Figure 7.9 The region in Example 3.

$\int_{-1.265}^{1.265} (2\cos x) - (x^2 - 1)\,dx$

$2\cos x = x^2 - 1$

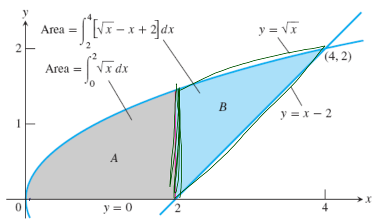
$2\int_{0}^{1.265} 2\cos x - (x^2 - 1)\,dx$

Boundaries with Changing Functions

If a boundary of a region is defined by more than one function, we can partition the region into subregions that correspond to the function changes and proceed as usual.

EXAMPLE 4 Finding Area Using Subregions

Find the area of the region R in the first quadrant that is bounded above by $y = \sqrt{x}$ and below by the x-axis and the line $y = x - 2$.

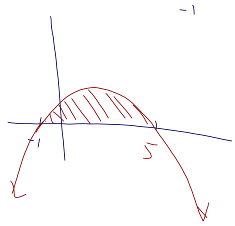


~~$\int_0^4 [\sqrt{x} - (x-2)]\,dx$~~

① $y = -x^2 + 6x - 5$

$(-x+1)(x-5)$

$-1 \quad 5$

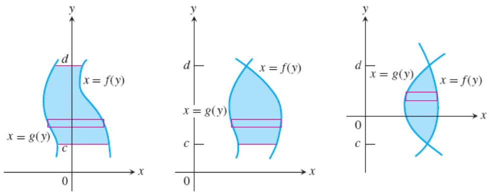


$\int_{-1}^{5} -x^2 + 6x - 5\,dx$

Integrating with Respect to y

Sometimes the boundaries of a region are more easily described by functions of y than by functions of x. We can use approximating rectangles that are horizontal rather than vertical and the resulting basic formula has y in place of x.

For regions like these



use this formula

$$A = \int_c^d [f(y) - g(y)]\,dy.$$

EXAMPLE 5 Integrating with Respect to y

Find the area of the region in Example 4 by integrating with respect to y.

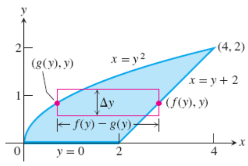
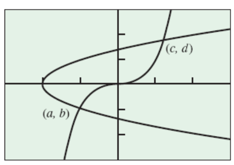


Figure 7.11 It takes two integrations to find the area of this region if we integrate with respect to x. It takes only one if we integrate with respect to y. (Example 5)

EXAMPLE 6 Making the Choice

Find the area of the region enclosed by the graphs of $y = x^3$ and $x = y^2 - 2$.



[–3, 3] by [–3, 3]

Figure 7.12 The region in Example 6.

EXAMPLE 7 Using Geometry

Find the area of the region in Example 4 by subtracting the area of the triangular region from the area under the square root curve.

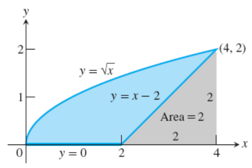


Figure 7.14 The area of the blue region is the area under the parabola $y = \sqrt{x}$ minus the area of the triangle. (Example 7)

Assignment:
page 395/ QRev 1 to 10
page 395/1 to 14 (all), STQ 50 to 55

Quick Review 7.2 *(For help, go to Sections 1.2 and 5.1.)*

In Exercises 1–5, find the area between the x-axis and the graph of the given function over the given interval.

1. $y = \sin x$ over $[0, \pi]$ 2

2. $y = e^{2x}$ over $[0, 1]$ $\frac{1}{2}(e^2 - 1) \approx 3.195$

3. $y = \sec^2 x$ over $[-\pi/4, \pi/4]$ 2

4. $y = 4x - x^3$ over $[0, 2]$ 4

5. $y = \sqrt{9 - x^2}$ over $[-3, 3]$ $9\pi/2$

In Exercises 6–10, find the x- and y-coordinates of all points where the graphs of the given functions intersect. If the curves never intersect, write "NI."

6. $y = x^2 - 4x$ and $y = x + 6$ (6, 12); (−1, 5)

7. $y = e^x$ and $y = x + 1$ (0, 1)

8. $y = x^2 - \pi x$ and $y = \sin x$ (0, 0); (π, 0)

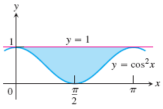
9. $y = \dfrac{2x}{x^2 + 1}$ and $y = x^3$ (−1, −1); (0, 0); (1, 1)

10. $y = \sin x$ and $y = x^3$ (−0.9286, −0.8008); (0, 0); (0.9286, 0.800

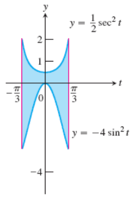
Section 7.2 Exercises

In Exercises 1–6, find the area of the shaded region analytically.

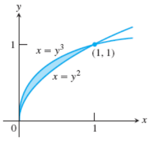
1. $\pi/2$



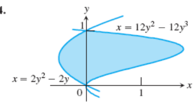
2. $4\pi/3$



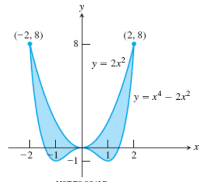
3. 1/12



4. 4/3

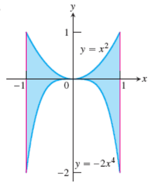


5. 128/15



NOT TO SCALE

6. 22/15

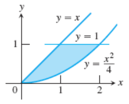


In Exercises 7 and 8, use a calculator to find the area of the region enclosed by the graphs of the two functions.

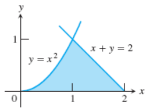
7. $y = \sin x, y = 1 - x^2$ ≈ 1.670 **8.** $y = \cos(2x), y = x^2 - 2$ ≈ 4.332

In Exercises 9 and 10, find the area of the shaded region analytically.

9. 5/6



10. 5/6

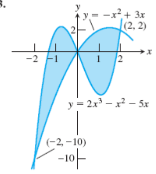


In Exercises 11 and 12, find the area enclosed by the graphs of the two curves by integrating with respect to y.

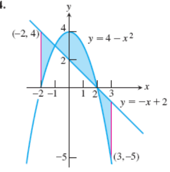
11. $y^2 = x + 1, y^2 = 3 - x$ ≈ 7.542 **12.** $y^2 = x + 3, y = 2x$ ≈ 7.146

In Exercises 13 and 14, find the total shaded area.

13. 16



14. $8\frac{1}{6}$



In Exercises 15–34, find the area of the regions enclosed by the lines and curves.

15. $y = x^2 - 2$ and $y = 2$ $10\frac{2}{3}$

16. $y = 2x - x^2$ and $y = -3$ $10\frac{2}{3}$

17. $y = 7 - 2x^2$ and $y = x^2 + 4$ 4

18. $y = x^4 - 4x^2 + 4$ and $y = x^2$ 8

19. $y = x\sqrt{a^2 - x^2}$, $a > 0$, and $y = 0$ $\frac{2}{3}a^3$

20. $y = \sqrt{|x|}$ and $5y = x + 6$ $1\frac{2}{3}$ (3 points of intersection)
(How many intersection points are there?)

21. $y = |x^2 - 4|$ and $y = (x^2/2) + 4$ $21\frac{1}{3}$

22. $x = y^2$ and $x = y + 2$ $4\frac{1}{2}$

23. $y^2 - 4x = 4$ and $4x - y = 16$ $30\frac{3}{8}$

24. $x - y^2 = 0$ and $x + 2y^2 = 3$ 4

25. $x + y^2 = 0$ and $x + 3y^2 = 2$ 8/3

26. $4x^2 + y = 4$ and $x^4 - y = 1$ $6\frac{14}{15}$

27. $x + y^2 = 3$ and $4x + y^2 = 0$ 8

28. $y = 2\sin x$ and $y = \sin 2x$, $0 \le x \le \pi$ 4

29. $y = 8\cos x$ and $y = \sec^2 x$, $-\pi/3 \le x \le \pi/3$ $6\sqrt{3}$

30. $y = \cos(\pi x/2)$ and $y = 1 - x^2$ $\frac{4}{3} - \frac{4}{\pi} \approx 0.0601$

31. $y = \sin(\pi x/2)$ and $y = x$ $\frac{4 - \pi}{\pi} \approx 0.273$

32. $y = \sec^2 x$, $y = \tan^2 x$, $x = -\pi/4$, $x = \pi/4$ $\frac{\pi}{2}$

33. $x = \tan^2 y$ and $x = -\tan^2 y$, $-\pi/4 \le y \le \pi/4$ $4 - \pi \approx 0.858$

34. $x = 3\sin y\sqrt{\cos y}$ and $x = 0$, $0 \le y \le \pi/2$ 2

In Exercises 35 and 36, find the area of the region by subtracting the area of a triangular region from the area of a larger region.

35. The region on or above the x-axis bounded by the curves $y^2 = x + 3$ and $y = 2x$. ≈ 4.333

36. The region on or above the x-axis bounded by the curves $y = 4 - x^2$ and $y = 3x$. 15/2

37. Find the area of the propeller-shaped region enclosed by the curve $x - y^3 = 0$ and the line $x - y = 0$. 1/2

38. Find the area of the region in the first quadrant bounded by the line $y = x$, the line $x = 2$, the curve $y = 1/x^2$, and the x-axis. 1

39. Find the area of the "triangular" region in the first quadrant bounded on the left by the y-axis and on the right by the curves $y = \sin x$ and $y = \cos x$. $\sqrt{2} - 1 \approx 0.414$

40. Find the area of the region between the curve $y = 3 - x^2$ and the line $y = -1$ by integrating with respect to **(a)** x, **(b)** y. 32/3

41. The region bounded below by the parabola $y = x^2$ and above by the line $y = 4$ is to be partitioned into two subsections of equal area by cutting across it with the horizontal line $y = c$.

(a) Sketch the region and draw a line $y = c$ across it that looks about right. In terms of c, what are the coordinates of the points where the line and parabola intersect? Add them to your figure. $(-\sqrt{c}, c); (\sqrt{c}, c)$

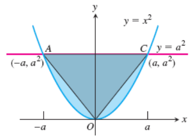
(b) Find c by integrating with respect to y. (This puts c in the limits of integration.) $\int_0^c \sqrt{y}\,dy = \int_c^4 \sqrt{y}\,dy \Rightarrow c = 2^{4/3}$

(c) Find c by integrating with respect to x. (This puts c into the integrand as well.)

42. Find the area of the region in the first quadrant bounded on the left by the y-axis, below by the line $y = x/4$, above left by the curve $y = 1 + \sqrt{x}$, and above right by the curve $y = 2/\sqrt{x}$. 11/3

41. (c) $\int_0^{\sqrt{c}} (c - x^2)\,dx = (4 - c)\sqrt{c} + \int_{\sqrt{c}}^2 (4 - x^2)\,dx \Rightarrow c = 2^{4/3}$

43. The figure here shows triangle AOC inscribed in the region cut from the parabola $y = x^2$ by the line $y = a^2$. Find the limit of the ratio of the area of the triangle to the area of the parabolic region as a approaches zero. 3/4

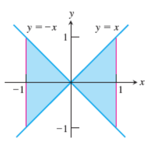


44. Suppose the area of the region between the graph of a positive continuous function f and the x-axis from $x = a$ to $x = b$ is 4 square units. Find the area between the curves $y = f(x)$ and $y = 2f(x)$ from $x = a$ to $x = b$. 4

45. Writing to Learn Which of the following integrals, if either, calculates the area of the shaded region shown here? Give reasons for your answer. Neither; both are zero

i. $\int_{-1}^{1} (x - (-x))\,dx = \int_{-1}^{1} 2x\,dx$

ii. $\int_{-1}^{1} (-x - (x))\,dx = \int_{-1}^{1} -2x\,dx$



46. Writing to Learn Is the following statement true, sometimes true, or never true? The area of the region between the graphs of the continuous functions $y = f(x)$ and $y = g(x)$ and the vertical lines $x = a$ and $x = b$ $(a < b)$ is

$$\int_a^b [f(x) - g(x)]\,dx.$$

Give reasons for your answer. Sometimes; If $f(x) \geq g(x)$ on (a, b), then true.

47. Find the area of the propeller-shaped region enclosed between the graphs of $\ln 4 - (1/2) \approx 0.886$

$$y = \frac{2x}{x^2 + 1} \quad \text{and} \quad y = x^3.$$

48. Find the area of the propeller-shaped region enclosed between the graphs of $y = \sin x$ and $y = x^3$. ≈ 0.4303

49. Find the positive value of k such that the area of the region enclosed between the graph of $y = k \cos x$ and the graph of $y = kx^2$ is 2. $k \approx 1.8269$

Standardized Test Questions

You should solve the following problems without using a graphing calculator.

50. True or False The area of the region enclosed by the graph of $y = x^2 + 1$ and the line $y = 10$ is 36. Justify your answer.
True. 36 is the value of the appropriate integral.

51. True or False The area of the region in the first quadrant enclosed by the graphs of $y = \cos x$, $y = x$, and the y-axis is given by the definite integral $\int_0^{0.739} (x - \cos x)\, dx$. Justify your answer. False. It is $\int_0^{0.739} (\cos x - x)\, dx$.

52. Multiple Choice Let R be the region in the first quadrant bounded by the x-axis, the graph of $x = y^2 + 2$, and the line $x = 4$. Which of the following integrals gives the area of R? A

(A) $\int_0^{\sqrt{2}} [4 - (y^2 + 2)]\, dy$ (B) $\int_0^{\sqrt{2}} [(y^2 + 2) - 4]\, dy$

(C) $\int_{-\sqrt{2}}^{\sqrt{2}} [4 - (y^2 + 2)]\, dy$ (D) $\int_{-\sqrt{2}}^{\sqrt{2}} [(y^2 + 2) - 4]\, dy$

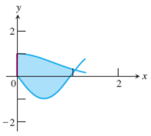
(E) $\int_2^4 [4 - (y^2 + 2)]\, dy$

53. Multiple Choice Which of the following gives the area of the region between the graphs of $y = x^2$ and $y = -x$ from $x = 0$ to $x = 3$? E

(A) 2 (B) 9/2 (C) 13/2 (D) 13 (E) 27/2

54. Multiple Choice Let R be the shaded region enclosed by the graphs of $y = e^{-x^2}$, $y = -\sin(3x)$, and the y-axis as shown in the figure below. Which of the following gives the approximate area of the region R? B

(A) 1.139 (B) 1.445 (C) 1.869 (D) 2.114 (E) 2.340



55. Multiple Choice Let f and g be the functions given by $f(x) = e^x$ and $g(x) = 1/x$. Which of the following gives the area of the region enclosed by the graphs of f and g between $x = 1$ and $x = 2$? A

(A) $e^2 - e - \ln 2$

(B) $\ln 2 - e^2 + e$

(C) $e^2 - \frac{1}{2}$

(D) $e^2 - e - \frac{1}{2}$

(E) $\frac{1}{e} - \ln 2$