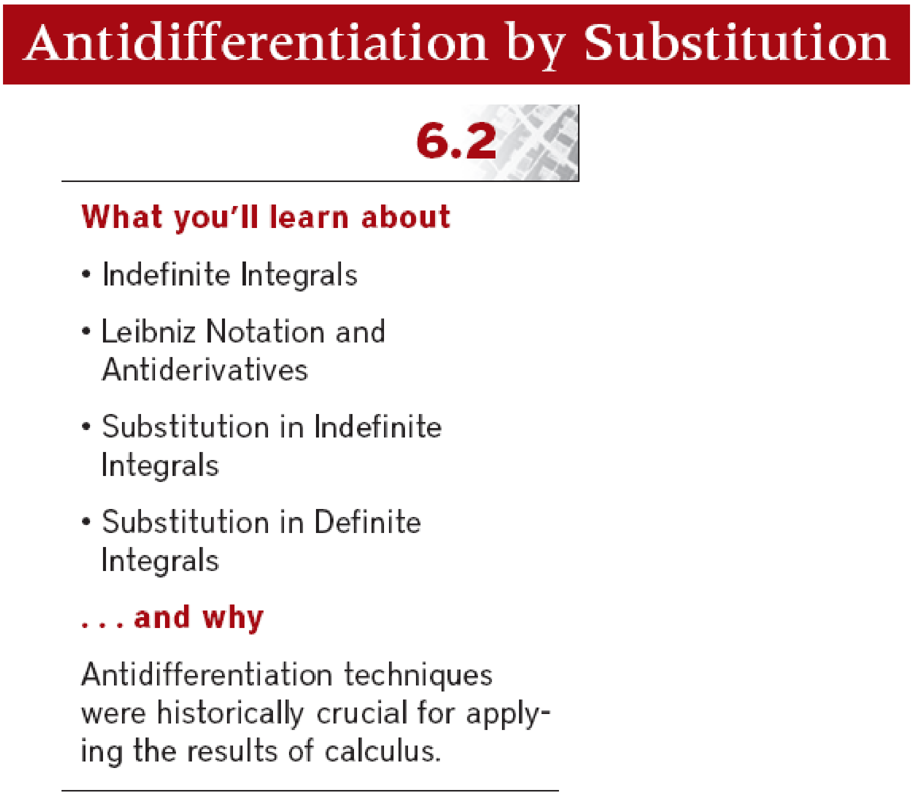


Antidifferentiation by Substitution

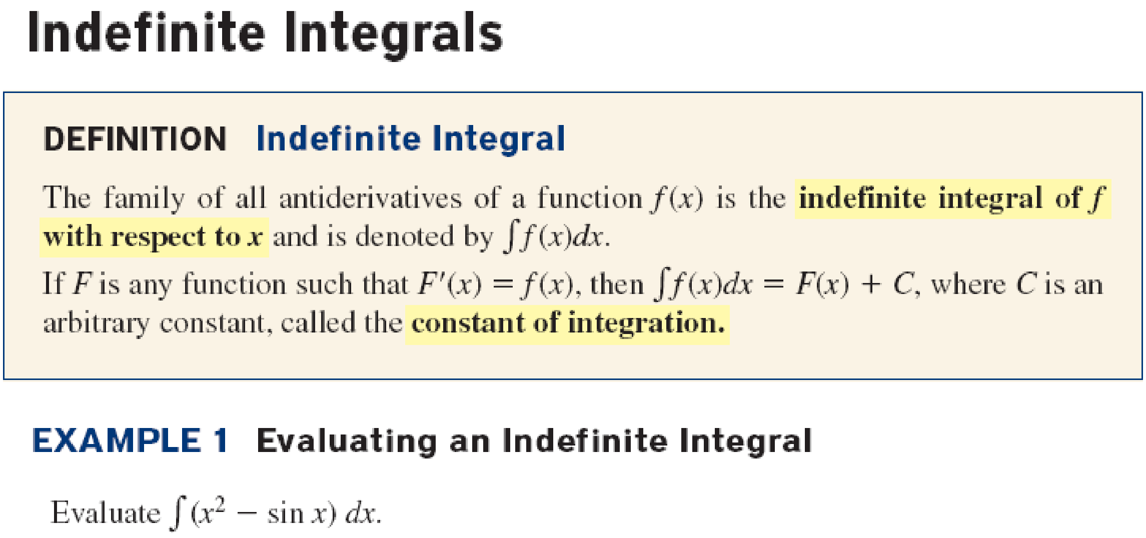
6.2

What you'll learn about

- Indefinite Integrals
- Leibniz Notation and Antiderivatives
- Substitution in Indefinite Integrals
- Substitution in Definite Integrals

. . . and why

Antidifferentiation techniques were historically crucial for applying the results of calculus.



Indefinite Integrals

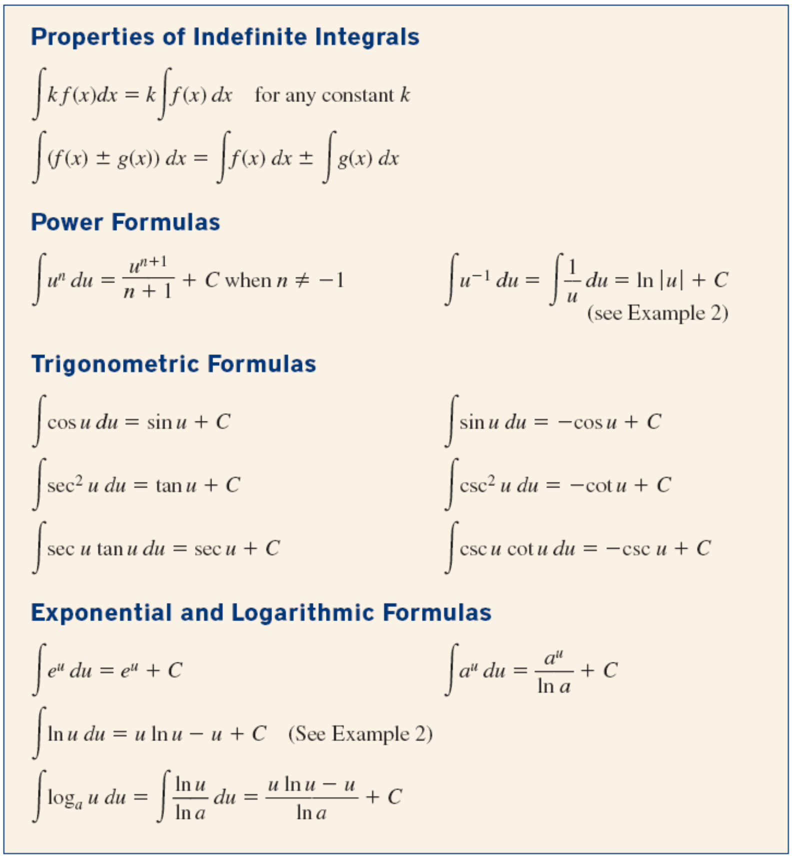
DEFINITION Indefinite Integral

The family of all antiderivatives of a function $f(x)$ is the **indefinite integral of f with respect to x** and is denoted by $\int f(x)dx$.

If F is any function such that $F'(x) = f(x)$, then $\int f(x)dx = F(x) + C$, where C is an arbitrary constant, called the **constant of integration.**

EXAMPLE 1 Evaluating an Indefinite Integral

Evaluate $\int (x^2 - \sin x)\, dx$.



Properties of Indefinite Integrals

$$\int k\,f(x)dx = k\int f(x)\,dx \quad \text{for any constant } k$$

$$\int (f(x) \pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx$$

Power Formulas

$$\int u^n\,du = \frac{u^{n+1}}{n+1} + C \text{ when } n \neq -1 \qquad \int u^{-1}\,du = \int \frac{1}{u}\,du = \ln|u| + C$$

(see Example 2)

Trigonometric Formulas

$$\int \cos u\,du = \sin u + C \qquad \int \sin u\,du = -\cos u + C$$

$$\int \sec^2 u\,du = \tan u + C \qquad \int \csc^2 u\,du = -\cot u + C$$

$$\int \sec u \tan u\,du = \sec u + C \qquad \int \csc u \cot u\,du = -\csc u + C$$

Exponential and Logarithmic Formulas

$$\int e^u\,du = e^u + C \qquad \int a^u\,du = \frac{a^u}{\ln a} + C$$

$$\int \ln u\,du = u\ln u - u + C \quad \text{(See Example 2)}$$

$$\int \log_a u\,du = \int \frac{\ln u}{\ln a}\,du = \frac{u\ln u - u}{\ln a} + C$$

EXAMPLE 2 Verifying Antiderivative Formulas

Verify the antiderivative formulas:

(a) $\int u^{-1}\,du = \int \frac{1}{u}\,du = \ln|u| + C$ (b) $\int \ln u\,du = u\ln u - u + C$

EXPLORATION 1 Are $\int f(u)\,du$ and $\int f(u)\,dx$ the Same Thing?

EXAMPLE 3 Paying Attention to the Differential

Let $f(x) = x^3 + 1$ and let $u = x^2$. Find each of the following antiderivatives in terms of x:

(a) $\int f(x)\,dx$ (b) $\int f(u)\,du$ (c) $\int f(u)\,dx$

Substitution in Indefinite Integrals

The Chain rule for antiderivatives...

EXAMPLE 4 Using Substitution

Evaluate $\int \sin x\, e^{\cos x}\,dx$.

EXAMPLE 5 Using Substitution

Evaluate $\int x^2\sqrt{5+2x^3}\,dx$.

EXAMPLE 6 Using Substitution

Evaluate $\int \cot 7x\,dx$.

EXAMPLE 7 Setting Up a Substitution with a Trigonometric Identity

Find the indefinite integrals. In each case you can use a trigonometric identity to set up a substitution.

(a) $\int \frac{dx}{\cos^2 2x}$ (b) $\int \cos^3 x\,dx = \int \cos^2 x \cos x\,dx$

$$\int \frac{1}{\cos^2(2x)}\,dx$$

$$\int \sec^2(2x)\,\overset{\frac{1}{2}du}{\cancel{dx}}$$

easy

not as easy

$$\frac{1}{2}\tan(2x)+C$$

$u = 2x$

$du = 2dx$

$\frac{1}{2}du = dx$

$$\frac{1}{2}\int \sec^2 u\,du$$

$$\frac{1}{2}\tan u + C$$

$$\frac{1}{2}\tan(2x)+C$$

$$= \int (1-\sin^2 x)\cos x\,dx$$

$u = \sin x$

$du = \cos x\,dx$

$$= \int 1-u^2\,du$$

$$= u - \frac{1}{3}u^3 + C$$

$$\sin x - \frac{1}{3}\sin^3 x + C$$

$$\int e^{3x}\,dx$$

$u = 3x$

$du = 3dx$

$\frac{1}{3}du = dx$

$$\frac{1}{3}e^{3x} + C$$

Substitution in Definite Integrals

EXAMPLE 8 **Evaluating a Definite Integral by Substitution**

Evaluate $\int_0^{\pi/3} \tan x \sec^2 x \, dx$.

$$\int_0^{\frac{\pi}{3}} (\tan x \sec x) \sec x \, dx$$

$u = \sec x$

$du = \sec x \tan x \, dx$

$u = \sec 0 = \frac{1}{\cos 0} = \frac{1}{1}$

$u = \sec \frac{\pi}{3} = \frac{1}{\cos \frac{\pi}{3}} = \frac{1}{\frac{1}{2}} = 2$

$$\int_1^2 u \, du$$

$$\frac{1}{2}u^2 \Big]_1^2$$

$$\frac{1}{2}(2)^2 - \frac{1}{2}(1)^2$$

$$2 - \frac{1}{2} = \boxed{\frac{3}{2}}$$

$$\int u \, du$$

$$\frac{1}{2}u^2$$

$$\frac{1}{2}(\sec x)^2 \Big]_0^{\frac{\pi}{3}}$$

EXAMPLE 9 **That Absolute Value Again**

Evaluate $\int_0^1 \frac{x}{x^2 - 4} \, dx.$

$u = x^2 - 4$

$du = 2x \, dx$

$\boxed{\frac{1}{2} du} = \boxed{x \, dx}$

$u = x^2 - 4$

$$\frac{1}{2} \int_{-4}^{-3} \frac{1}{u} \, du$$

$$\frac{1}{2} \ln|u| \Big]_{-4}^{-3}$$

$$\frac{1}{2} \left(\ln|-3| - \ln|-4| \right)$$

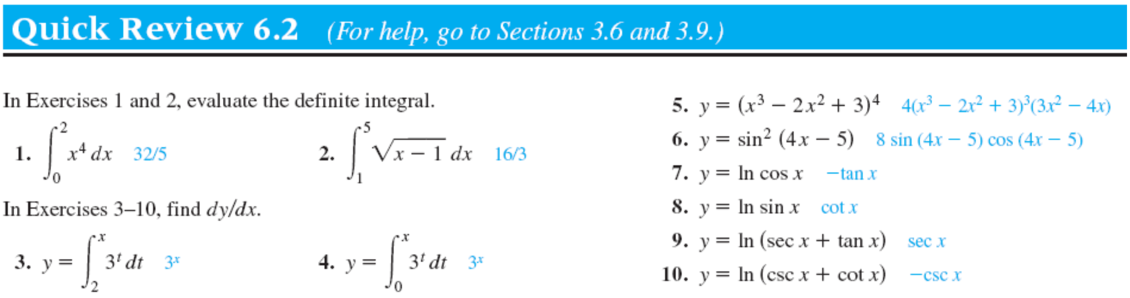
$$\frac{1}{2} (\ln 3 - \ln 4)$$

$$\boxed{\frac{1}{2} \left(\ln\left(\frac{3}{4}\right) \right)}$$

$$\boxed{\ln\left(\sqrt{\frac{3}{4}}\right)}$$

Assignment #1:
page 337/ Quick Review 1 to 10
page 337/ 1 to 29 odd

Assignment #2:
page 338/ 31 to 43 odd
53 to 67 odd
STQ, 71 to 76



Quick Review 6.2 *(For help, go to Sections 3.6 and 3.9.)*

In Exercises 1 and 2, evaluate the definite integral.

1. $\int_0^2 x^4\,dx$ 32/5

2. $\int_1^5 \sqrt{x-1}\,dx$ 16/3

In Exercises 3–10, find dy/dx.

3. $y = \int_2^x 3^t\,dt$ 3^x

4. $y = \int_0^x 3^t\,dt$ 3^x

5. $y = (x^3 - 2x^2 + 3)^4$ $4(x^3 - 2x^2 + 3)^3(3x^2 - 4x)$

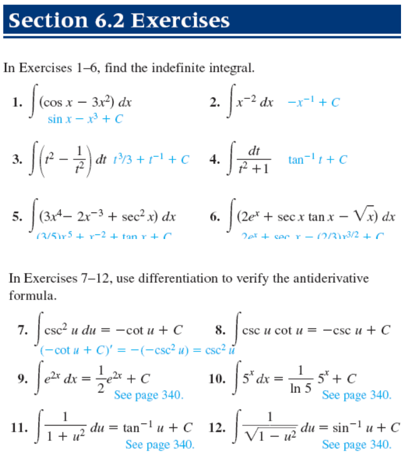
6. $y = \sin^2(4x - 5)$ $8\sin(4x - 5)\cos(4x - 5)$

7. $y = \ln \cos x$ $-\tan x$

8. $y = \ln \sin x$ $\cot x$

9. $y = \ln(\sec x + \tan x)$ $\sec x$

10. $y = \ln(\csc x + \cot x)$ $-\csc x$



Section 6.2 Exercises

In Exercises 1–6, find the indefinite integral.

1. $\int (\cos x - 3x^2)\,dx$ $\sin x - x^3 + C$

2. $\int x^{-2}\,dx$ $-x^{-1} + C$

3. $\int \left(t^2 - \frac{1}{t^2}\right) dt$ $t^3/3 + t^{-1} + C$

4. $\int \frac{dt}{t^2 + 1}$ $\tan^{-1} t + C$

5. $\int (3x^4 - 2x^{-3} + \sec^2 x)\,dx$ $(3/5)x^5 + x^{-2} + \tan x + C$

6. $\int (2e^x + \sec x \tan x - \sqrt{x})\,dx$ $2e^x + \sec x - (2/3)x^{3/2} + C$

In Exercises 7–12, use differentiation to verify the antiderivative formula.

7. $\int \csc^2 u\,du = -\cot u + C$ $(-\cot u + C)' = -(-\csc^2 u) = \csc^2 u$

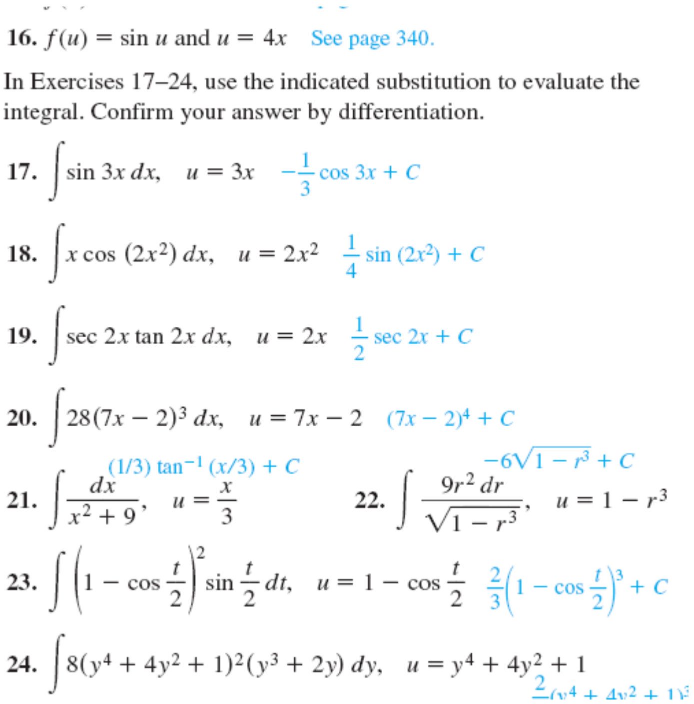
8. $\int \csc u \cot u = -\csc u + C$

9. $\int e^{2x}\,dx = \frac{1}{2}e^{2x} + C$ See page 340.

10. $\int 5^x\,dx = \frac{1}{\ln 5}5^x + C$ See page 340.

11. $\int \frac{1}{1 + u^2}\,du = \tan^{-1} u + C$ See page 340.

12. $\int \frac{1}{\sqrt{1 - u^2}}\,du = \sin^{-1} u + C$ See page 340.



16. $f(u) = \sin u$ and $u = 4x$ See page 340.

In Exercises 17–24, use the indicated substitution to evaluate the integral. Confirm your answer by differentiation.

17. $\int \sin 3x\,dx, \quad u = 3x$ $-\frac{1}{3}\cos 3x + C$

18. $\int x \cos(2x^2)\,dx, \quad u = 2x^2$ $\frac{1}{4}\sin(2x^2) + C$

19. $\int \sec 2x \tan 2x\,dx, \quad u = 2x$ $\frac{1}{2}\sec 2x + C$

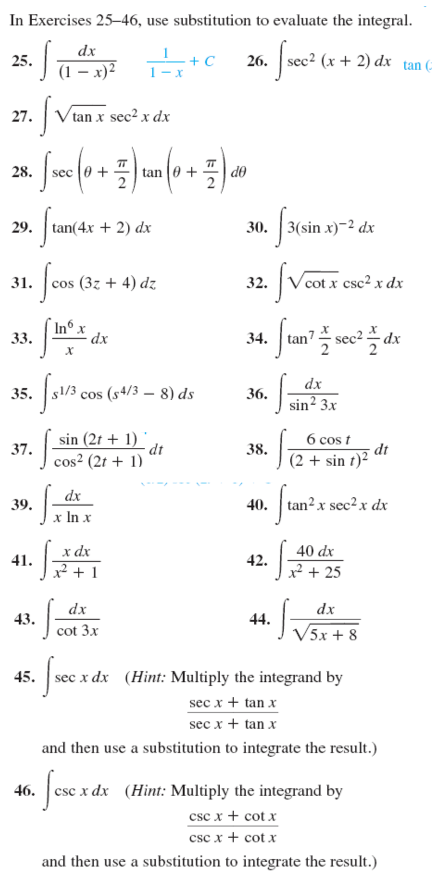
20. $\int 28(7x - 2)^3\,dx, \quad u = 7x - 2$ $(7x - 2)^4 + C$

21. $\int \frac{dx}{x^2 + 9}, \quad u = \frac{x}{3}$ $(1/3)\tan^{-1}(x/3) + C$

22. $\int \frac{9r^2\,dr}{\sqrt{1 - r^3}}, \quad u = 1 - r^3$ $-6\sqrt{1 - r^3} + C$

23. $\int \left(1 - \cos\frac{t}{2}\right)^2 \sin\frac{t}{2}\,dt, \quad u = 1 - \cos\frac{t}{2}$ $\frac{2}{3}\left(1 - \cos\frac{t}{2}\right)^3 + C$

24. $\int 8(y^4 + 4y^2 + 1)^2(y^3 + 2y)\,dy, \quad u = y^4 + 4y^2 + 1$ $\frac{2}{}(y^4 + 4y^2 + 1)$



In Exercises 25–46, use substitution to evaluate the integral.

25. $\int \frac{dx}{(1-x)^2}$ $\frac{1}{1-x}+C$ **26.** $\int \sec^2(x+2)\,dx$ tan (

27. $\int \sqrt{\tan x}\,\sec^2 x\,dx$

28. $\int \sec\left(\theta+\frac{\pi}{2}\right)\tan\left(\theta+\frac{\pi}{2}\right)d\theta$

29. $\int \tan(4x+2)\,dx$ **30.** $\int 3(\sin x)^{-2}\,dx$

31. $\int \cos(3z+4)\,dz$ **32.** $\int \sqrt{\cot x}\,\csc^2 x\,dx$

33. $\int \frac{\ln^6 x}{x}\,dx$ **34.** $\int \tan^7\frac{x}{2}\sec^2\frac{x}{2}\,dx$

35. $\int s^{1/3}\cos(s^{4/3}-8)\,ds$ **36.** $\int \frac{dx}{\sin^2 3x}$

37. $\int \frac{\sin(2t+1)}{\cos^2(2t+1)}\,dt$ **38.** $\int \frac{6\cos t}{(2+\sin t)^2}\,dt$

39. $\int \frac{dx}{x\ln x}$ **40.** $\int \tan^2 x\sec^2 x\,dx$

41. $\int \frac{x\,dx}{x^2+1}$ **42.** $\int \frac{40\,dx}{x^2+25}$

43. $\int \frac{dx}{\cot 3x}$ **44.** $\int \frac{dx}{\sqrt{5x+8}}$

45. $\int \sec x\,dx$ (*Hint:* Multiply the integrand by

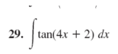
$$\frac{\sec x+\tan x}{\sec x+\tan x}$$

and then use a substitution to integrate the result.)

46. $\int \csc x\,dx$ (*Hint:* Multiply the integrand by

$$\frac{\csc x+\cot x}{\csc x+\cot x}$$

and then use a substitution to integrate the result.)



29. $\int \tan(4x+2)\,dx$

$\int \frac{\sin(4x+2)}{\cos(4x+2)}$

$u=4x+2$

$du=4\,dx$

$\frac{1}{4}du=dx$

$\frac{1}{4}\int \tan u\,du$

$\frac{1}{4}\int \frac{\sin u}{\cos u}\,du$ let $w=\cos u$

$dw=-\sin u\,du$

$-dw=\sin u\,du$

$\frac{1}{4}\int \frac{1}{\cos u}\boxed{\sin u\,du}$

$-\frac{1}{4}\int \frac{1}{w}\,dw$

$-\frac{1}{4}\ln w+C$

$-\frac{1}{4}\ln(\cos u)+C$

$-\frac{1}{4}\ln(\cos(4x+2))+C$

$\cancel{-\frac{1}{4}}\frac{1}{\cos(4x+2)}\cdot\cancel{-}\sin(4x+2)\cdot\cancel{4}$

27. $\int \sqrt{\tan x}\, \sec^2 x\, dx$

$u = \tan x$

$du = \sec^2 x\, dx$

$\int u^{\frac{1}{2}}\, du$

$\frac{2}{3} u^{3/2} + C$

In Exercises 47–52, use the given trigonometric identity to set up a u-substitution and then evaluate the indefinite integral.

47. $\int \sin^3 2x\, dx, \quad \sin^2 2x = 1 - \cos^2 2x$

48. $\int \sec^4 x\, dx, \quad \sec^2 x = 1 + \tan^2 x$

49. $\int 2 \sin^2 x\, dx, \quad \cos 2x = 2 \sin^2 x - 1$

50. $\int 4 \cos^2 x\, dx, \quad \cos 2x = 1 - 2 \cos^2 x$

51. $\int \tan^4 x\, dx, \quad \tan^2 x = \sec^2 x - 1$

52. $\int (\cos^4 x - \sin^4 x) dx, \quad \cos 2x = \cos^2 x - \sin^2 x$

47. $\int \sin^3 2x\,dx$, $\underline{\sin^2 2x = 1 - \cos^2 2x}$

$$\int \sin(2x)\cdot \sin^2(2x)\,dx$$

$$\int \underline{\sin(2x)}\cdot\left(1-\overset{(\cos 2x)^2}{\cos^2 2x}\right)\underline{dx}$$

$u = \cos(2x)$

$du = -2\sin(2x)\,dx$

$-\frac{1}{2}\,du = \sin(2x)\,dx$

$$-\frac{1}{2}\int(1-u^2)\,du$$

$$-\frac{1}{2}\left(u-\frac{1}{3}u^3\right)+C$$

$$-\frac{1}{2}\left(\cos(2x)-\frac{1}{3}(\cos(2x))^3\right)+C$$

In Exercises 53–66, make a u-substitution and integrate from $u(a)$ to $u(b)$.

53. $\int_0^3 \sqrt{y+1}\,dy$

54. $\int_0^1 r\sqrt{1-r^2}\,dr$

55. $\int_{-\pi/4}^0 \tan x\sec^2 x\,dx$

56. $\int_{-1}^1 \frac{5r}{(4+r^2)^2}\,dr$

57. $\int_0^1 \frac{10\sqrt{\theta}}{(1+\theta^{3/2})^2}\,d\theta$

58. $\int_{-\pi}^{\pi} \frac{\cos x}{\sqrt{4+3\sin x}}\,dx$

59. $\int_0^1 \sqrt{t^5+2t}\,(5\cancel{t^4+2)\,dt}$

60. $\int_0^{\pi/6} \cos^{-3}2\theta\,\sin 2\theta\,d\theta$

61. $\int_0^7 \frac{dx}{x+2}$

62. $\int_2^5 \frac{dx}{2x-3}$

63. $\int_1^2 \frac{dt}{t-3}$

64. $\int_{\pi/4}^{3\pi/4} \cot x\,dx$

65. $\int_{-1}^3 \frac{x\,dx}{x^2+1}$

66. $\int_0^2 \frac{e^x\,dx}{3+e^x}$

(59) $u = t^5+2t$

$du = 5t^4+2\,dt$

$$\int_0^3 \sqrt{u}\,du$$

$$\frac{2}{3}u^{\frac{3}{2}}\Big]_0^3$$

Two Routes to the Integral In Exercises 67 and 68, make a substitution $u = \cdots$ (an expression in x), $du = \cdots$. Then

(a) integrate with respect to u from $u(a)$ to $u(b)$.

(b) find an antiderivative with respect to u, replace u by the expression in x, then evaluate from a to b.

67. $\displaystyle\int_0^1 \frac{x^3}{\sqrt{x^4+9}}\,dx$

68. $\displaystyle\int_{\pi/6}^{\pi/3} (1-\cos 3x)\sin 3x\,dx$

69. Show that

$$y = \ln\left|\frac{\cos 3}{\cos x}\right| + 5$$

is the solution to the initial value problem

$$\frac{dy}{dx} = \tan x, \quad f(3) = 5.$$

(See the discussion following Example 4, Section 5.4.)

70. Show that

$$y = \ln\left|\frac{\sin x}{\sin 2}\right| + 6$$

is the solution to the initial value problem

$$\frac{dy}{dx} = \cot x, \quad f(2) = 6.$$

Standardized Test Questions

You should solve the following problems without using a graphing calculator.

71. True or False By u-substitution, $\int_0^{\pi/4} \tan^3 x \sec^2 x\,dx = \int_0^{\pi/4} u^3\,du$. Justify your answer.

72. True or False If f is positive and differentiable on $[a, b]$, then $\displaystyle\int_a^b \frac{f'(x)dx}{f(x)} = \ln\left(\frac{f(b)}{f(a)}\right)$. Justify your answer.

73. Multiple Choice $\int \tan x\,dx =$

(A) $\dfrac{\tan^2 x}{2} + C$

(B) $\ln|\cot x| + C$

(C) $\ln|\cos x| + C$

(D) $-\ln|\cos x| + C$

(E) $-\ln|\cot x| + C$

74. Multiple Choice $\int_0^2 e^{2x}\,dx =$

(A) $\dfrac{e^4}{2}$ **(B)** $e^4 - 1$ **(C)** $e^4 - 2$ **(D)** $2e^4 - 2$ **(E)** $\dfrac{e^4 - 1}{2}$

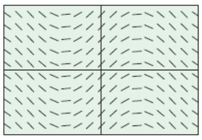
75. Multiple Choice If $\int_3^5 f(x-a)\,dx = 7$ where a is a constant, then $\int_{3-a}^{5-a} f(x)\,dx =$

(A) $7 + a$ **(B)** 7 **(C)** $7 - a$ **(D)** $a - 7$ **(E)** -7

76. Multiple Choice If the differential equation $dy/dx = f(x)$ leads to the slope field shown below, which of the following could be $\int f(x)\,dx$?

(A) $\sin x + C$ **(B)** $\cos x + C$ **(C)** $-\sin x + C$

(D) $-\cos x + C$ **(E)** $\dfrac{\sin^2 x}{2} + C$



Explorations

77. ***Constant of Integration*** Consider the integral

$$\int \sqrt{x+1}\,dx.$$

(a) Show that $\int \sqrt{x+1}\,dx = \frac{2}{3}(x+1)^{3/2} + C.$

(b) **Writing to Learn** Explain why

$$y_1 = \int_0^x \sqrt{t+1}\,dt \quad \text{and} \quad y_2 = \int_3^x \sqrt{t+1}\,dt$$

are antiderivatives of $\sqrt{x+1}$.

(c) Use a table of values for $y_1 - y_2$ to find the value of C for which $y_1 = y_2 + C$.

(d) **Writing to Learn** Give a convincing argument that

$$C = \int_0^3 \sqrt{x+1}\,dx.$$

78. **Group Activity** ***Making Connections*** Suppose that

$$\int f(x)\,dx = F(x) + C.$$

(a) Explain how you can use the derivative of $F(x) + C$ to confirm the integration is correct.

(b) Explain how you can use a slope field of f and the graph of $y = F(x)$ to support your evaluation of the integral.

(c) Explain how you can use the graphs of $y_1 = F(x)$ and $y_2 = \int_0^x f(t)\,dt$ to support your evaluation of the integral.

(d) Explain how you can use a table of values for $y_1 - y_2$, y_1 and y_2 defined as in part (c), to support your evaluation of the integral.

(e) Explain how you can use graphs of f and NDER of $F(x)$ to support your evaluation of the integral.

(f) Illustrate parts (a)–(e) for $f(x) = \dfrac{x}{\sqrt{x^2+1}}$.