# 9.2: Taylor Series

Brook Taylor was an accomplished musician and painter. He did research in a variety of areas, but is most famous for his development of ideas regarding infinite series.

**Brook Taylor**
**1685 - 1731**

Greg Kelly, Hanford High School, Richland, Washington

Suppose we wanted to find a fourth degree polynomial of the form:

$$P(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$$

that approximates the behavior of $f(x) = \ln(x+1)$ at $x = 0$

If we make $P(0) = f(0)$, and the first, second, third and fourth derivatives the same, then we would have a pretty good approximation.

$\rightarrow$

$$P(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$$

$$f(x) = \ln(x+1)$$

$$f(x) = \ln(x+1) \qquad P(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$$

$$f(0) = \ln(1) = 0 \qquad P(0) = a_0 \longrightarrow \boxed{a_0 = 0}$$

$$f'(x) = \frac{1}{1+x} \qquad P'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3$$

$$f'(0) = \frac{1}{1} = 1 \qquad P'(0) = a_1 \longrightarrow \boxed{a_1 = 1}$$

$$f''(x) = -\frac{1}{(1+x)^2} \qquad P''(x) = 2a_2 + 6a_3 x + 12a_4 x^2$$

$$f''(0) = -\frac{1}{1} = -1 \qquad P''(0) = 2a_2 \longrightarrow \boxed{a_2 = -\frac{1}{2}}$$

$\rightarrow$

$$P(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$$

$$f(x) = \ln(x+1)$$

$$f''(x) = -\frac{1}{(1+x)^2}$$

$$P''(x) = 2a_2 + 6a_3 x + 12a_4 x^2$$

$$f''(0) = -\frac{1}{1} = -1$$

$$P''(0) = 2a_2 \rightarrow \boxed{a_2 = -\frac{1}{2}}$$

$$f'''(x) = 2 \cdot \frac{1}{(1+x)^3}$$

$$P'''(x) = 6a_3 + 24a_4 x$$

$$f'''(0) = 2$$

$$P'''(0) = 6a_3 \rightarrow \boxed{a_3 = \frac{2}{6}}$$

$$f^{(4)}(x) = -6\frac{1}{(1+x)^4}$$

$$P^{(4)}(x) = 24a_4$$

$$f^{(4)}(0) = -6$$

$$P^{(4)}(0) = 24a_4 \rightarrow \boxed{a_4 = -\frac{6}{24}}$$

$\rightarrow$

$$P(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$$

$$f(x) = \ln(x+1)$$

$$P(x) = 0 + 1x - \frac{1}{2}x^2 + \frac{2}{6}x^3 - \frac{6}{24}x^4$$

$$P(x) = 0 + x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$$

$$f(x) = \ln(x+1)$$

If we plot both functions, we see that near zero the functions match very well!

$f(x)$

$P(x)$

Our polynomial: $$0+1x-\frac{1}{2}x^2+\frac{2}{6}x^3-\frac{6}{24}x^4$$

has the form: $$f(0)+f'(0)x+\frac{f''(0)}{2}x^2+\frac{f'''(0)}{6}x^3+\frac{f^{(4)}(0)}{24}x^4$$

or: $$\frac{f(0)}{0!}+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\frac{f^{(4)}(0)}{4!}x^4$$

This pattern occurs no matter what the original function was!

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## Maclaurin Series:

(generated by $f$ at $x=0$ )

$$P(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\ \cdots$$

If we want to center the series (and it's graph) at some point other than zero, we get the Taylor Series:

## Taylor Series:

(generated by $f$ at $x=a$ )

$$P(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3+\ \cdots$$

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example: $y = \cos x$

$$f(x) = \cos x \qquad f(0) = 1$$

$$f'(x) = -\sin x \qquad f'(0) = 0$$

$$f''(x) = -\cos x \qquad f''(0) = -1$$

$$f'''(x) = \sin x \qquad f'''(0) = 0$$

$$f^{(4)}(x) = \cos x \qquad f^{(4)}(0) = 1$$

$$P(x) = 1 + 0x - \frac{1x^2}{2!} + \frac{0x^3}{3!} + \frac{1x^4}{4!} + \frac{0x^5}{5!} - \frac{1x^6}{6!} + \cdots$$

$$\boxed{P(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} \cdots}$$

$\rightarrow$

$$y = \cos x$$

$$P(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} \cdots$$

The more terms we add, the better our approximation.

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example: $y = \cos(2x)$

Rather than start from scratch, we can use the function that we already know:

$$P(x) = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \frac{(2x)^8}{8!} - \frac{(2x)^{10}}{10!} \quad \cdots$$

$\rightarrow$

example: $y=\cos(x)$ at $x=\dfrac{\pi}{2}$

$$f(x)=\cos x \quad f\left(\frac{\pi}{2}\right)=0$$

$$f'(x)=-\sin x \quad f'\left(\frac{\pi}{2}\right)=-1$$

$$f''(x)=-\cos x \quad f''\left(\frac{\pi}{2}\right)=0$$

$$f'''(x)=\sin x \quad f'''\left(\frac{\pi}{2}\right)=1$$

$$f^{(4)}(x)=\cos x \quad f^{(4)}\left(\frac{\pi}{2}\right)=0$$

$$P(x)=0-1\left(x-\frac{\pi}{2}\right)+\frac{0}{2!}\left(x-\frac{\pi}{2}\right)^2+\frac{1}{3!}\left(x-\frac{\pi}{2}\right)^3+\cdots$$

$$\boxed{P(x)=-\left(x-\frac{\pi}{2}\right)+\frac{\left(x-\frac{\pi}{2}\right)^3}{3!}-\frac{\left(x-\frac{\pi}{2}\right)^5}{5!}+\cdots}$$

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When referring to Taylor polynomials, we can talk about **number of terms**, **order** or **degree**.

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$$

This is a polynomial in **3 terms**.

It is a **4th order** Taylor polynomial, because it was found using the 4th derivative.

It is also a **4th degree** polynomial, because $x$ is raised to the 4th power.

The **3rd order** polynomial for $\cos x$ is $1 - \frac{x^2}{2!}$, but it is **degree 2**.

A recent AP exam required the student to know the difference between *order* and *degree*.

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## The TI-89 finds Taylor Polynomials:

taylor (expression, variable, order, [point])

$\underbrace{\text{F3} \quad 9}$

**F3** **9**

taylor $\left(\cos(x), x, 6\right)$ $\qquad \dfrac{-x^6}{720}+\dfrac{x^4}{24}-\dfrac{x^2}{2}+1$

taylor $\left(\cos(2x), x, 6\right)$ $\qquad \dfrac{-4x^6}{45}+\dfrac{2x^4}{3}-2x^2+1$

taylor $\left(\cos(x), x, 5, \pi/2\right)$ $\qquad \dfrac{-(2x-\pi)^5}{3840}+\dfrac{(2x-\pi)^3}{48}-\dfrac{2x-\pi}{2}$

$\pi$